Unit 2: Word Problems Involving Geometry

It is assumed that the reader has an understanding of the essay titled “Introduction to Modeling”.

Making Sketches

It is almost always helpful to make a sketch when trying to solve a problem which involves geometric shapes. The sketch need not be elaborate, detailed, realistic, or even to scale. The purpose of the sketch is to organize the information and the question into a visual and familiar geometric version. It is one step in the translation process. Make the sketch only as accurate, detailed, and realistic as is necessary to accomplish that goal.

Sometimes the sketches are actual drawings of the situation and in other cases the sketches may simply be diagrams (Venn, hierarchy, flow, tree, network, maps, mind map, etc.) to show relationships. Recall the following non stipulative definition of diagram:

A diagram is a two-dimensional geometric symbolic representation of information according to some visualization technique.

Writing in Mathematics

The presentation of the following examples and analysis is designed to accomplish several goals. The student should learn to present word problems and their solutions in a manner quite similar to the illustrations shown below. The “writing” must be part of the presentation. Most students incorrectly believe that the computations are the mathematics. On the other hand mathematics professors and other professionals recognize that the logic, reasoning, organization, and writing is the mathematics.

As a student it is important to keep in mind that your task is not to find the answer, but rather your task is to:

i. Learn problem solving techniques.
ii. Convince your instructor that you understand a particular technique.
iii. Convince your instructor that you can apply a particular technique in a variety of situations.

Observe the use of the word “technique” as opposed to the word “process”. You should strive to understand the technique rather than memorizing an “x-step process” for each different problem type. For example, the application of The Transitive Property for Equality is the central most important technique in the solution of any word problem. Making a sketch when geometric shapes are involved is an important and quite general problem solving technique. These and other problem solving techniques will be discussed when they are used in examples.

As a professional faced with a non-trivial problem (math or otherwise) in the workplace you will be expected to communicate your analysis and solution of the problem to other people. The organization of such communication is extremely important. An analysis will be understood by the intended audience only if it is well organized. The examples presented here are an initial step in the process of learning the necessary organizational skills.
Example 1: The length of a rectangle is 4 units more than its width and the perimeter of the rectangle is 8 times the width. What is the width of the rectangle?

Analysis: Refer to Figure 1.
Let \( x \) be the width of the rectangle.
Then the length is \( x + 4 \).
According to the statement of the problem the perimeter \( P \) is 8\( x \).
According to Figure 1 the perimeter is 2\( x \) + 2\( (x + 4) \).
We now have two expressions for the perimeter.
Therefore (by the Transitive Property) the two expressions must be equal.
This observation yields the model 2\( x \) + 2\( (x + 4) \) = 8\( x \).
Solving this equation will produce the width.
\[
2x + 2(x + 4) = 8x \\
2x + 2x + 8 = 8x \\
4x - 8x + 8 = 0 \\
-4x = -8 \\
x = 2
\]
The rectangle is 2 units wide.

Example 2: A rectangular parcel of land is 50 ft. wide. The length of a diagonal between opposite corners is 10 ft. more than the length of the parcel. What is the length of the parcel?

Analysis: Refer to Figure 2.
Let \( x \) be the length of the rectangle.
Then the length of the diagonal is \( x + 10 \).
The Pythagorean Theorem informs us that the length of the diagonal is \( \sqrt{x^2 + (50)^2} \).
We now have two expressions for the length of the diagonal.
Therefore (by the Transitive Property) the two expressions must be equal.
This observation yields the model \( \sqrt{x^2 + (50)^2} = x + 10 \).
Solving this equation will produce the length.
\[
\sqrt{x^2 + (50)^2} = x + 10 \\
x^2 + (50)^2 = (x + 10)^2 \\
x^2 + 2500 = x^2 + 20x + 100 \\
2400 = 20x \\
x = 120
\]
The length of the rectangle is 120 ft.
Example 3: A square plot of land contains a 50 ft. long by 30 ft. wide building. The rest of the land outside the building forms a parking lot. The area of the parking lot is 9,000 sq. ft. What are the dimensions of the plot of land?

Analysis: Refer to Figure 3.
Let \( x \) be the width of the square parcel.
The area of the parking lot is 9,000. (Try to find another expression for this area).
Clearly the area of the parking lot is the area of the entire parcel minus the area of the building.
The area of the parcel is \( x^2 \).
The area of the building is \( (50)(30) = 1500 \).
Therefore the area of the parking lot is \( x^2 - 1500 \)
We now have two expressions for the area of the parking lot.
Therefore (by the Transitive Property) the two expressions must be equal.
This observation yields the model \( x^2 - 1500 = 9000 \)
Solving this equation will produce the width of the square parcel.
\[
x^2 - 1500 = 9000
x^2 = 10500
x = \pm \sqrt{10500} = \pm \sqrt{(100)(105)} = \pm 10\sqrt{105}
\]
Because width is a positive number, we discard the negative solution.
The exact width of the square parcel is \( 10\sqrt{105} \).
The approximate width of the square parcel, correct to one decimal place, is \( 10(10.25) = 102.5 \).

Example 4: A gallon of latex paint can cover 300 square feet. How many gallon containers of paint should be purchased to paint each wall of a rectangular room which is 15 feet long, 12 feet wide, and 8 feet high?

Analysis: Refer to Figure 4.
Let \( x \) be the required number of gallons.
Then \( x = \frac{A}{300} \) where \( A \) is the total area to be painted.
The area of two side walls is \( 2)(15)(8) = 240 \)
The area of two end walls is \( 2)(12)(8) = 192 \)
The total area to be painted is \( A = 240 + 192 = 432 \)
Then \( x = \frac{432}{300} = 1.44 \)
The assumption is that this paint is only available in gallons, so the painter must purchase 2 gallons of paint.
Example 5: The base of a triangle is 24 inches long. The area of the triangle is 60 square inches. What is the height of the triangle?

Analysis: Refer to Figure 5.
Let $h$ be the height of the triangle.
The area of the triangle is 60
The area of the triangle is $\frac{1}{2}(24)h = 12h$
We now have two expressions for the area of the triangle.
Therefore (by the Transitive Property) the two expressions must be equal.
This observation yields the model $12h = 60$
Solving this equation will produce the height of the triangle.
$12h = 60$
$h = 5$
The triangle is 5 inches high.

Example 6: The length of one of the legs of a right triangle is 5 cm. If the area of the triangle is 10 cm, what is the length of the other leg?

Analysis: Refer to Figure 6.
Let $x$ be the length of the other leg.
The area of the triangle is 10
The area of the triangle is $\frac{1}{2}(5)x = \frac{5}{2}x$
We now have two expressions for the area of the triangle.
Therefore (by the Transitive Property) the two expressions must be equal.
This observation yields the model $\frac{5}{2}x = 10$
Solving this equation will produce the desired length.
$\frac{5}{2}x = 10$
$x = \frac{2}{5}(10) = 4$
The length of the other leg is 4 cm.

Example 7: The length of one of the bases of a trapezoid is one more than twice that of the other base. The altitude is 2 in. If the area of the trapezoid is 19, what are the lengths of its bases?

Analysis: Refer to Figure 7.
Let $x$ be the length of the shortest base.
Then the length of the longer base is $2x + 1$.
The area of the trapezoid is $A = 19$. 

---

C:\Users\De\Dropbox\myMathematics\WordDocuments\Modelling\Modeling_Word_Problems_Geometry.doc
Using the formula \( A = \frac{1}{2} (B + b)h \) for the area of a trapezoid we obtain

\[
A = \frac{1}{2} (x + (2x + 1))(2) = 3x + 1
\]

We now have two expressions for the area \( A \) of the trapezoid. Therefore (by the Transitive Property) the two expressions must be equal. This observation yields the model \( 3x + 1 = 19 \)

Solving this equation will produce the desired length.

\[
3x + 1 = 19
\]

\[
3x = 18
\]

\[
x = 6
\]

The short base of the trapezoid is 6 inches long

The long base of the trapezoid is 13 inches.

**Example 8:** A cone has radius 3 feet and volume \( 36\pi \) cubic feet. What is the height?

**Analysis:** Refer to Figure 8.

Let \( h \) be the height of the cone.

The volume \( V = 36\pi \)

Using the formula \( V = \frac{1}{3} \pi r^2 h \) for the volume of a cone we obtain

\[
V = \frac{1}{3} \pi (9)h = 3\pi h
\]

We now have two expressions for the volume of the cone. Therefore (by the Transitive Property) the two expressions must be equal. This observation yields the model \( 3\pi h = 36\pi \)

Solving this equation will produce the desired length.

\[
3\pi h = 36\pi \\
\]

\[
h = 12
\]

The height of the cone is 12 inches.

**Example 9:** What is the height of a cone shaped can with radius 2.5 inches and volume 40 cu. in.?

**Analysis:** Refer to Figure 9.

Let \( h \) be the height of the cone.

The volume is 40

Using the formula \( V = \frac{1}{3} \pi r^2 h \) for the volume of a cone we obtain

\[
V = \frac{1}{3} \pi \left(\frac{25}{4}\right)h = \frac{25}{12} \pi h \quad \text{(note } 2.5 = \frac{5}{2} \text{)}
\]

We now have two expressions for the volume of the cone. Therefore (by the Transitive Property) the two expressions must be equal. This observation yields the model \( \frac{25}{12} \pi h = 40 \)
Solving this equation will produce the desired length.
\[
\frac{25}{12} \pi h = 40
\]
\[
h = \frac{(12)(40)}{25\pi} = \frac{(12)(8)}{5\pi} = \frac{96}{5\pi}
\]
The height of the cone is \( \frac{96}{5\pi} \) inches.
The approximate height, correct to two decimal places, is 6.11 inches.

**Example 10:** A storage bin for corn consists of a cylindrical section made of wire mesh, surmounted by a conical tin roof. The radius of the structure is 10 feet and the height of the conical tin roof is one-third the total height of the structure. The volume of storage bin is \(1400\pi\) cu. ft. What is the height of the cylindrical part?

**Analysis:** Refer to Figure 10.

Let \( h \) be the total height of the structure.

Then the conical top has height \( \frac{h}{3} \) and the cylindrical section has height \( \frac{2h}{3} \).

The total volume of the bin is \( V_{\text{bin}} = 1400\pi \). (Try to find another expression for total volume)

The volume of the bin is the volume of the cylindrical part plus the volume of the conical part.

Using the formula \( V = \pi r^2 h \) for the volume of a cylinder we get
\[
V_{\text{cyl}} = \pi (10^2) \left( \frac{2}{3} h \right) = \frac{200}{3} \pi h
\]

Using the formula \( V = \frac{1}{3} \pi r^2 h \) for the volume of a cone we get
\[
V_{\text{cone}} = \left( \frac{1}{3} \right) \pi (10^2) \left( \frac{h}{3} \right) = \frac{100}{9} \pi h
\]

When the volume of the cylindrical part and the conical part are added we obtain
\[
V_{\text{bin}} = V_{\text{cyl}} + V_{\text{cone}} = \frac{200}{3} \pi h + \frac{100}{9} \pi h = \frac{700}{9} \pi h
\]

We now have two expressions for the total volume. Therefore (by the Transitive Property) the two expressions must be equal.

This observation yields the model \( \frac{700}{9} \pi h = 1400\pi \)

Solving this equation for \( h \) will produce the desired height.
\[
\frac{700}{9} \pi h = 1400\pi
\]
\[
h = \left( \frac{9}{700\pi} \right) (1400\pi) = 18
\]

The total height of the structure is 18 feet
The height of the cylindrical part is two-thirds the total height or 12 feet.
Example 11: A can manufacturer has a contract to make cylindrical cans with a radius of 3 inches and a volume of $15\pi$ cubic inches. What should be the height of the cans?

Analysis: Refer to Figure 11.

Known values can be substituted into the formula \( V = \pi r^2 h \) for the volume of a cylinder to obtain \( 15\pi = \pi(9)h \)

Which can be solved for \( h \) to obtain \( h = \frac{5}{3} \) feet as the height of the cans.

Example 12: The diameter of a cylindrical propane gas tank is 4 feet. The volume of the tank is 603.2 cubic feet. Find the length of the tank.

Analysis: Refer to Figure 12.

Let \( x \) be the length of the cylinder. The volume of the tank is \( V = 603.2 \)

Using the formula for the volume \( V = \pi r^2 h \) of a cylinder we get \( V = \pi (2)^2 h = 16\pi h \)

We now have two expressions for the volume. Therefore (by the Transitive Property) the two expressions must be equal. This observation yields the model \( 16\pi h = 603.2 \)

Solving this equation will produce the length of the cylinder.

\[
16\pi h = 603.2 \\
16(3.14)h = 603.2 \\
h = 12.00
\]

The cylinder is 12 feet long.

Note that in this problem we used 3.14 as an approximation for \( \pi \) and we used a calculator to do the divisions.

Example 13: If a cylinder has a radius of 5 ft. its surface area, including the two ends, is \( 500\pi \) sq. ft., what is its height?

Analysis: Refer to Figure 13.

Let \( h \) be the height of the cylinder.

The surface area of the cylinder is \( A_S = 500\pi \) . (Try to find another expression for this surface area.)

The surface area is the area of the two ends plus the area of the curved surface.

The area of the two ends is \( A_E = 2\left(\pi r^2\right) = 2(25\pi) = 50\pi \) .

The area of the curved surface is \( A_C = 2\pi rh = 2\pi(5)h = 10\pi h \)

The Surface area of the cylinder is \( A_c = 50\pi + 10\pi h \)

We now have two expressions for the surface area. Therefore (by the Transitive Property) the two expressions must be equal. This observation yields the model \( 50\pi + 10\pi h = 500\pi \)
Solving this equation will produce the length of the cylinder.

\[ 50\pi + 10\pi h = 500\pi \]

\[ 10\pi h = 500\pi - 50\pi = 450\pi \]

\[ h = \frac{450\pi}{10\pi} = 45 \]

The cylinder is 45 feet tall.

Note that it was advantageous to work with \( \pi \) rather than a numerical approximation.

**Example 14:** A farmer wants to build two adjacent corrals using exactly 100 yards of fencing. The two corrals are to have the same dimensions. He wants the combined area of the two corrals to be 350 square yards. What are the dimensions of the corrals?

**Analysis:** Refer to Figure 14.

Let \( x \) be the width of a corral.
Let \( y \) be the length of a corral.

The length of fencing used is 100.

According to Figure 14 the length of fencing is \( 4x + 3y \).

We have two expressions for the length of fencing.

Therefore (by the Transitive Property) the two expressions must be equal.

This observation yields \( 4x + 3y = 100 \)

Because there are two variables, it is necessary to construct another equation. This second equation must be independent of the first. Consider the information about the area.

The area is 350

According to Figure 14, the area is \( 2xy \)

We have two expressions for the area.

Therefore (by the Transitive Property) the two expressions must be equal.

This observation yields \( 2xy = 350 \) which is equivalent to \( xy = 175 \)

The model for this problem is a system of equations and the solution to the problem will be a solution for that system.

The system of equations in this case is

\[
\begin{align*}
4x + 3y & = 100 \\
x\cdot y & = 175
\end{align*}
\]

A simple system like this is best solved using the substitution method.

Solving the first equation for \( y \) produces the equivalent equation \( y = \frac{100 - 4x}{3} \)

That value for \( y \) may be substituted into the second equation to obtain

\[ x \left( \frac{100 - 4x}{3} \right) = 175 \]

This equation can be solved to obtain a value (or values) for \( x \).
The quadratic formula can be used to solve this quadratic equation.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{-100 \pm \sqrt{100^2 - (4)(4)(525)}}{2(4)} = \frac{100 \pm \sqrt{100^2 - (100)(4)(21)}}{8}
\]

\[
= \frac{100 \pm 40}{8} = \frac{140}{8} \approx 17.5 \text{ or } \frac{60}{8} \approx 7.5
\]

There are two possible values for \( x \). Each of these values will determine a corresponding value of \( y \).

Remember that \( y = \frac{100 - 4x}{3} \).

Therefore if \( x = 17.5 \), then \( y = \frac{100 - 4(17.5)}{3} = 10 \)

Similarly if \( x = 7.5 \), then \( y = \frac{100 - 4(7.5)}{3} = 23.3 \)

In one case the dimensions for each corral are 17.5 and 10 while in the other case the dimensions are 7.5 and 23.3.

Mathematics produces two suitable solutions. The farmer must then choose one or the other.

Corrals measuring 7.5 by 23.3 might be fine for a dog run, but for restraining larger livestock, the farmer will probably choose corrals which measure 17.5 by 10.

**Example 15:** A rose garden measuring 12 ft. by 7 ft. is surrounded by a grass border of equal width. The total area of rose garden and border is 300 sq. ft. What is the width of the grass border?

**Analysis:** Refer to Figure 15.

Let \( x \) be the width of the grass border.

The area of the entire parcel is 300.

The area of the entire parcel is \((2x + 7)(2x + 12)\).

We now have two expressions for the area of the entire parcel.

Therefore (by the Transitive Property) the two expressions must be equal.

This observation yields the model \( 4x^2 + 38x + 84 = 300 \)

Solving this equation will produce the width of the border.


4x^2 + 38x + 84 = 300
4x^2 + 38x - 216 = 0
2x^2 + 19x - 108 = 0
(x - 4)(2x + 27) = 0

By the Zero Factor Property
x - 4 = 0 or 2x + 27 = 0
x = 4 or x = \(-\frac{27}{2}\)

Because the width must be a positive number the only possibility is x = 4.
The grass border is 4 feet wide.

**Example 16:** A trough is 12 ft. long, 3 ft. wide and 3 ft. deep. What is the depth of water when the trough contains 70 gallons of water? Hint: One cubic foot is approximately 7.48100 gallons.

**Analysis:** Refer to Figure 17.
Let x be the depth of water.
The volume of water is (3)(12)x cu. ft.
The volume of water is 70 gallons. (this must be converted to cu. ft.)
1 cu. ft. = 7.481 gal.

Then 1 gal. \(\approx \frac{1}{7.481}\) cu. ft.

Therefore 70 gal. \(\approx \frac{70}{7.481}\) \(\approx 9.357\) cu. ft.

So the volume of water is approximately 9.357 cu. ft.
We now have two expressions for the volume of water.
This observation yields the model \((3)(12)x \approx 9.357\)
This is the model for the problem.
Solving this equation will produce the desired dimension.

\((3)(12)x \approx 9.357\)
\[x \approx \frac{9.357}{36} \approx 0.2599 \text{ ft.} \approx 3.12 \text{ in.}\]
The water is about 3.12 inches deep.

**Example 17:** A winch is used to tow a boat to a dock. The rope is attached to the boat at a point 15 feet below the level of the winch. How far is the boat from the dock when 75 feet of rope is out?

**Analysis:** Refer to Figure 17.
Let x be the distance from boat to dock.

Then according to The Pythagorean Theorem \(x^2 + 15^2 = 75^2\)
This is the model for the problem and the solutions to this equation will answer the question.
\[ x^2 + 15^2 = 75^2 \]
\[ x^2 = 75^2 - 15^2 = 15^2(5 - 1) = 15^2(2^2) = 30^2 \]
\[ x = \pm 30 \]
Distance must be positive, so we conclude the boat is 30 feet from the dock.

**Example 18:** An open box is to be made from a square sheet of cardboard by cutting out 3 inch squares from each corner and then folding up the sides. The box must have a volume of 300 cubic inches. What must be the dimensions of the original sheet of cardboard?

**Analysis:** Refer to Figure 18.
Let \( x \) be the width of the large square.
The volume of the box is 300
The volume of the box is
\[ 3(x - 6)(x - 6) = 3(x^2 - 12x + 36) \]
We now have two expressions for the volume of the box. Therefore (by the Transitive Property) the two expressions must be equal. This observation yields the model \( 3(x^2 - 12x + 36) = 300 \)
This is the model for the problem and the solutions to this equation will answer the question. Solving this equation will produce the desired dimension.
\[ 3(x^2 - 12x + 36) = 300 \]
\[ x^2 - 12x + 36 = 100 \]
\[ x^2 - 12 - 64 = 0 \]
\[ (x - 16)(x + 4) = 0 \]
By The Zero Factor Property
\[ x - 16 = 0 \text{ or } x + 4 = 0 \]
\[ x = 16 \text{ or } x = -4 \]
Because the width must be a positive number the only possibility is \( x = 16 \).
The original square must be 16 inches wide.

**Example 19:** A rectangle is three times longer than it is wide. Its diagonal is 50 inches long. What are the dimensions of the rectangle? What is its perimeter? What is its area?

**Analysis:** Refer to Figure 19.
Let \( x \) be the width of the rectangle.
Then its length is \( 3x \).
According to the Pythagorean Theorem \((3x)^2 + x^2 = 50^2 \)
This is the model for the problem and the solutions to this equation will answer the question.
(3x)^2 + x^2 = 50^2
10x^2 = 2500
x^2 = 250
x = ±√250 = ±√(25)(10) = ±5√10
Length must be positive, so we conclude the rectangle is
5√10 inches wide and 15√10 inches long.
The area of the rectangle is A = (5√10)(15√10) = (5)(15)(10) = 750 sq. in.
The perimeter of the rectangle is P = 2(5√10 + 15√10) = 2(20√10) = 40√10 inches.

Example 20: A post 4 feet tall makes a 3-foot shadow. If the shadow
of a brick wall is 22.5 feet long, how tall is the wall?
Analysis: Refer to Figure 20.
Let x be the height of the wall.
Recognize that the two triangles are similar triangles.
Therefore \( \frac{x}{22.5} = \frac{4}{3} \)
This is the model for the problem and the solutions to this equation
will answer the question.
\[
\frac{x}{22.5} = \frac{4}{3} \\
x = \frac{(22.5)(4)}{3} = \frac{90}{3} = 30
\]
The brick wall is 30 feet tall.

Example 21: A garden measuring 13 meters by 9 meters has a
pedestrian pathway around it, increasing the total area
to 285 square meters. Find the width of the pathway?
Analysis: Refer to Figure 21.
Let x be the width of the grass border.
The area of the entire parcel is 285.
The area of the entire parcel is (2x + 13)(2x + 9).
We now have two expressions for the area of the entire parcel.
Therefore (by the Transitive Property) the two expressions must be
equal.
This observation yields the model \( 4x^2 + 44x + 117 = 285 \)
Solving this equation will produce the width of the border.
\[4x^2 + 44x + 117 = 285\]
\[4x^2 + 44x - 168 = 0\]
\[x^2 + 11x - 42 = 0\]
\[(x - 3)(x + 14) = 0\]
By The Zero Factor Property
\[x - 3 = 0 \quad \text{or} \quad x + 14 = 0\]
\[x = 3 \quad \text{or} \quad x = -14\]
Because the width must be a positive number the only possibility is \(x = 3\).
The border is 3 feet wide.

**Example 22:** Consider the drawing in Figure 22. Calculate the width of the river.

**Analysis:** Refer to Figure 22
Let \(x\) be the width of the river.
The triangles ACD and DEF are similar triangles.
Therefore
\[\frac{x + 10}{20} = \frac{70}{55}\]
This is the model for the problem.
Solving this equation will produce the width of the river.
\[\frac{x + 10}{20} = \frac{70}{55}\]
\[x + 10 = \frac{(20)(70)}{55} = \frac{(4)(70)}{11} = \frac{280}{11}\]
\[x = \frac{280}{11} - 10 = \frac{280 - 110}{11} = \frac{170}{11} \approx 15.45\]
The river is about 15.45 feet wide.

**Example 23:** Two vertical poles of lengths 6 feet and 8 feet stand 10 feet apart. A cable reaches from the top of one pole to some anchor point on the ground between the poles and then to the top of the other pole. Where should this anchor point be located to use 18 feet of cable?

**Analysis:** Refer to Figure 23
Let \(x\) be the distance from the 6 foot pole to the anchor.
Then \(10 - x\) is the distance from the anchor to the 8 foot pole.
Let \(d\) be the segment of the cable from the 6 foot pole to the anchor.
Let \(m\) be the segment of the cable from the 8 foot pole to the anchor.
Then, according to the statement of the problem, \(d + m = 18\).
Application of the Pythagorean Theorem to the obvious two triangles yields:
We have two expressions for the quantity \( d + m \) so those two expressions must be equal. Therefore the model for this problem is:

\[
\sqrt{x^2 + 36} + \sqrt{x^2 - 20x + 164} = 18
\]

Solving this equation will determine the location of the anchor point.

Begin the process of solving this equation by squaring both sides. Keep in mind that when both sides are squared, the new equation might not be equivalent to the original. Looking ahead, it is clear we will need to use the squaring process two times.

\[
\sqrt{x^2 + 36} + \sqrt{x^2 - 20x + 164} = 18
\]

\[
(x^2 + 36) + (x^2 - 20x + 164) + 2\sqrt{x^2 + 36x^2 - 20x + 164} = 18^2
\]

\[
2\sqrt{x^2 + 36}x^2 - 20x + 164 = -2x^2 + 20x + 124
\]

\[
\sqrt{x^2 + 36}x^2 - 20x + 164 = -x^2 + 10x + 62
\]

\[
(x^2 + 36)(x^2 - 20x + 164) = (-x^2 + 10x + 62)^2
\]

\[
x^4 - 20x^3 + 164x^2 + 36x^2 - 720x + 5904 = x^4 - 20x^3 - 24x^2 + 1240x + 3844
\]

\[
x^2 - 1960x + 2060 = 0
\]

\[
56x^2 - 490x + 515 = 0
\]

Use the quadratic formula \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) to solve that quadratic equation. The use of a calculator is anticipated.

\[
56x^2 - 490x + 515 = 0
\]

\[
x = \frac{490 \pm \sqrt{(490)^2 - 4(56)(515)}}{2(56)} = \frac{490 \pm \sqrt{240100 - 115360}}{112} = \frac{490 \pm 353.19}{112}
\]

This yields the following two possibilities:

\[
x = \frac{490 + 353.19}{112} = 7.5 \quad \text{or} \quad x = \frac{490 - 353.19}{112} = 1.2
\]

Recall that during the process we generated equations which might not have been equivalent to the previous equation, so these two solutions must be tested in the original equation. However, because both solutions are physically possible, that is a sufficient test.

Therefore the anchor may be located 7.5 feet from the six foot pole or it can be located 1.2 feet from the six foot pole.

**Example 24:** A goat is tied to the corner of a 12-by-15 foot building with a rope 10 ft. long. What is the area the goat may graze?
**Analysis:**
Sketch a diagram which models the problem. The diagram need not be to scale, but the dimensions must be correct and it must accurately model the problem. The diagram may contain more detail than is required to analyze this problem.

It is presumed that the angles at the corners A, B, C, and D of the shed are each 90°. From the assumption that angle A is 90°, it follows that angle θ is 270° which yields three quarters of a circle.

We have just solved the problem! The area that the goat may graze is three-quarters of the area of a 10 ft. circle. Now all we need is a few computations.

Let A be the area grazed by the goat. Then

\[ A = \frac{3}{4} \pi r^2 = \frac{3}{4} \pi (10)^2 \]

\[ \frac{3}{4} \pi 100 = 75\pi \text{ sq. ft.} \]

The exact area that the goat may graze.

To the nearest sq. ft. the area is 236 sq. ft.

**Example 25:** A goat is tied to the corner of a 12-by-15 foot building with a rope 20 ft. long. What is the area the goat may graze?

**Analysis:**
In view of Example 24, we might be tempted to believe the area is three-quarters of the area of a 20 ft. circle. However, the sketch shown in at the right demonstrates that a simple change in the length of the tether significantly changes the problem.

The desired area still involves three-quarters of the area of a 20 ft. circle, but it also includes one-quarter of the area of a 5 ft. circle as well as one-quarter of the area of an 8 ft. circle.

Now all we need is a few computations.

Let A be the total area the goat may graze.

Let AM be three-quarters of the area of a 20 ft. circle (labeled M)

Let AN be one-quarter of the area of a 5 ft. circle (labeled N)

Let AP be one-quarter of the area of a 8 ft. circle (labeled P)

Then \( A = AM + AN + AP \) is a formula for the desired area.

The computations should be constructed and presented as indicated by this formula.
\[ A = A_M + A_N + A_P = \frac{3}{4}\pi (20)^2 + \frac{1}{4}\pi (5)^2 + \frac{1}{4}\pi (8)^2 \]
\[ = \frac{3}{4}\pi (400) + \frac{1}{4}\pi (25) + \frac{1}{4}\pi (64) \]
\[ = \frac{1200}{4}\pi + \frac{25}{4}\pi + \frac{64}{4}\pi = \frac{1289}{4}\pi \text{ sq. ft.} \]

The goat may graze exactly \(\frac{1289}{4}\pi\) sq. ft.

To the nearest sq. ft. the grazing area is 1012 sq. ft.