Linear vs Rational Equations in One Variable

Consider and solve the equation $\mathbf{\frac{x}{2} = \frac{21}{10} - \frac{x}{5}}$

The first questions must be: What kind of equation is this? Is it a rational equation? Is it a polynomial equation? If it is a polynomial equation, what is its degree? How many variables are in the equation? Where is its graph? Is the graph on the number line or in the Cartesian plane?

The fact that there are fractions in the equation makes it a rational equation. The fact that the only fractions are numbers (no variables in a denominator) makes it a polynomial equation.

This is a linear equation (a polynomial equation) in one variable because we can rewrite it as
$\frac{x}{2} + \frac{x}{5} - \frac{21}{10} = 0$ and then by adding like terms we get the equation $\left(\frac{7}{10}\right)x + \left(-\frac{21}{10}\right) = 0$
which is clearly of the linear form ax + b = 0.

Having determined that we are trying to solve a linear equation in one variable we know that only two tools are required. Those two tools are:

Properties of Equations:

(1) If any expression is added to both sides of an equation the resulting equation is equivalent to the original equation.

(2) If both sides of an equation are multiplied by the same non-zero real number, the resulting equation is equivalent to the original equation.

To solve $\left(\frac{7}{10}\right)x + \left(-\frac{21}{10}\right) = 0$ (which is equivalent to the original equation. Why?) add $\frac{21}{10}$ to both sides and then multiply both sides by $\frac{10}{7}$ to obtain the simplest equation x = 3 whose solution set is {3}.

In the above analysis each conversion of an equation to another equation used only the two **Properties of Equations** listed above. Therefore each equation (including the simplest one) is equivalent to the original equation.
Therefore the solution set for the original equation is the set {3}.

Because the equation is an equation in one variable, its graph is on the Real Number Line. The graph consists of a single dot at the number 3 on the number line.

Consider the equation $\frac{x+5}{x+3} = \frac{2}{x+3}$

The first questions must be: What kind of equation is this? Is it a rational equation? Is it a polynomial equation? If it is a polynomial equation, what is its degree? How many variables are in the equation? Where is its graph? Is the graph on the number line or in the Cartesian plane?

The fact that there are fractions with variables in the denominator in the equation makes it a rational equation which is not a polynomial equation. Clearly there is only one variable.

Having determined that we are trying to solve a rational equation in one variable we know that the two previously listed properties of equations, although still helpful, are not sufficient to solve the equation. We must employ another technique.

> Multiply both sides of the equation by the least common denominator. It contains a variable. Therefore the new equation might not be equivalent to the original equation. However, **the solution set of the original equation is a subset of the solution set of the new equation**.

Begin with the original equation

Eq. 1: $\frac{x+5}{x+3} = \frac{2}{x+3}$ Multiply both sides by x + 3 Suppose A is the solution set for Eq.1

Eq. 2: x + 5 = 2 Add – 5 to both sides Suppose B is the solution set for Eq.2. Then we know that $A \subset B$

Eq. 3: x = -3 Eq. 3 is equivalent to Eq. 2 so they have the same solution sets.

The solution set for Eq 3. is {-3}

Now we know that $A \subset B$ = {-3}.

At this point we know that -3 is the only possible solution. To determine the solution set for Eq. 1 we must test -3 in Eq. 1. If a true statement results, -3 is a solution. If a false, or meaningless, statement results, -3 is not a solution.

Important Fact: For rational equations, the only check that is required is to determine if the possible solution causes a zero in any denominator of the original equation.

For this particular Eq. 1, -3 causes both denominators to be zero. Consequently -3 is not a solution. Since -3 is the only possible solution, the solution set for Eq. 1 is the empty set $\varnothing$.

Observe that for rational equations in one variable, the test of all possible solutions is an important and integral part of the solution process. This is significantly different than for linear equations in one variable where the test is not required. This significant difference is caused by the fact that when both sides of an equation are multiplied by an

expression containing a variable, the resulting equation need not be equivalent to the original equation.

Consider the equation $\frac{\mathbf{1}}{\mathbf{x-1}} = \frac{\mathbf{2}}{\mathbf{x+1}}$

The first questions must be: What kind of equation is this? Is it a rational equation? Is it a polynomial equation? If it is a polynomial equation, what is its degree? How many variables are in the equation? Where is its graph? Is the graph on the number line or in the Cartesian plane?

The fact that there are fractions with variables in the denominator in the equation makes it a rational equation which is not a polynomial equation. Clearly there is only one variable. Having determined that we are trying to solve a rational equation in one variable we know that the two previously listed properties of equations, although still helpful, are not sufficient to solve the equation. We must employ another technique.

> Multiply both sides of the equation by the least common denominator. It contains a variable. Therefore the new equation might not be equivalent to the original equation. However, **the solution set of the original equation is a subset of the solution set of the new equation**.

Begin with the original equation

Eq. 1: $\frac{1}{x-1} = \frac{2}{x+1}$	Multiply both sides by $(x + 1)(x - 1)$	Suppose A is the solution set for Eq.1
Eq. 2: $x + 1 = 2x - 2$	Add $-(2x - 2)$ to both sides and multiply both sides by -1	Suppose B is the solution set for Eq.2. Then we know that $A \subset B$

Eq. 3: $x = 3$ Eq. 3 is equivalent to Eq. 2 so they have the same solution sets.

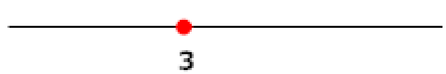
The solution set for Eq 3. is {3}

Now we know that $A \subset B = \{3\}$.

At this point we know that 3 is the only possible solution. To determine the solution set for Eq. 1 we must test 3 in Eq. 1. If a true statement results, 3 is a solution. If a false, or meaningless, statement results, 3 is not a solution.

Important Fact: For rational equations, the only check that is required is to determine if the possible solution causes a zero in any denominator of the original equation.

For this particular Eq. 1, 3 does not cause a denominator to be zero. Consequently 3 is a solution. Since 3 is the only possible solution, the solution set for Eq. 1 is the set {3}. Because the equation is an equation in one variable, its graph is on the Real Number Line. The graph consists of a single dot at the number 3 on the number line.



Observe that for rational equations in one variable, the test of all possible solutions is an important and integral part of the solution process. This is significantly different than for linear equations in one variable where the test is not required. This significant difference is caused by the fact that when both sides of an equation are multiplied by an

expression containing a variable, the resulting equation need not be equivalent to the original equation.

Consider the equation $\frac{2}{x^2-4}=\frac{1}{2x-4}$

The first questions must be: What kind of equation is this? Is it a rational equation? Is it a polynomial equation? If it is a polynomial equation, what is its degree? How many variables are in the equation? Where is its graph? Is the graph on the number line or in the Cartesian plane?

The fact that there are fractions with variables in the denominator in the equation makes it a rational equation which is not a polynomial equation. Clearly there is only one variable.

Having determined that we are trying to solve a rational equation in one variable we know that the two previously listed properties of equations, although still helpful, are not sufficient to solve the equation. We must employ another technique.

> Multiply both sides of the equation by the least common denominator. It contains a variable. Therefore the new equation might not be equivalent to the original equation. However, **the solution set of the original equation is a subset of the solution set of the new equation**.

Begin with the original equation

Eq. 1: $\frac{2}{x^2-4}=\frac{1}{2x-4}$ Multiply both sides by $2(x-2)(x+2)$ Suppose A is the solution set for Eq.1

Eq. 2: $2(2x-4)=x^2-4$ Add $-(4x-8)$ to both sides Suppose B is the solution set for Eq.2. Then we know that $A \subset B$

Eq. 3: $x^2-4x+4=0$ Eq. 3 is equivalent to Eq. 2 so they have the same solution sets.

$(x-2)^2=0$

The solution set for Eq 3. is {2} Now we know that $A \subset B = \{2\}$.

At this point we know that 2 is the only possible solution. To determine the solution set for Eq. 1 we must test 2 in Eq. 1. If a true statement results, 2 is a solution. If a false, or meaningless, statement results, 2 is not a solution.

For this particular Eq. 1, 2 causes both denominators to be zero. Consequently 2 is not a solution. Since 2 is the only possible solution, the solution set for Eq. 1 is the empty set $\varnothing$.

Observe that for rational equations in one variable, the test of all possible solutions is an important and integral part of the solution process. This is significantly different than for linear equations in one variable where the test is not required. This significant difference is caused by the fact that when both sides of an equation are multiplied by an

expression containing a variable, the resulting equation need not be equivalent to the original equation.

Consider the equation $\frac{x^2-23}{2x^2-5x-3}+\frac{2}{x-3}=\frac{-1}{2x+1}$

The first questions must be: What kind of equation is this? Is it a rational equation? Is it a polynomial equation? If it is a polynomial equation, what is its degree? How many variables are in the equation? Where is its graph? Is the graph on the number line or in the Cartesian plane?

The fact that there are fractions with variables in the denominator in the equation makes it a rational equation which is not a polynomial equation. Clearly there is only one variable. Having determined that we are trying to solve a rational equation in one variable we know that the two previously listed properties of equations, although still helpful, are not sufficient to solve the equation. We must employ another technique.

> Multiply both sides of the equation by the least common denominator. It contains a variable. Therefore the new equation might not be equivalent to the original equation. However, **the solution set of the original equation is a subset of the solution set of the new equation**.

However, before performing this multiplication it is wise to factor as many of the numerators and denominators as possible.

Start with the original equation

Eq. 1: $$\frac{x^2-23}{2x^2-5x-3}+\frac{2}{x-3}=\frac{-1}{2x+1}$$

and factor the denominator of the first fraction on the left side to obtain an equivalent equation.

Eq. 2: $$\frac{x^2-23}{(2x+1)(x-3)}+\frac{2}{x-3}=\frac{-1}{2x+1}$$

It is clear that the Least Common Denominator (LCD) is the product $(2x + 1)(x - 3)$.
Multiply both sides of Eq.2 to obtain

Eq. 3: $(x^2 - 23) + 2(2x + 1) = (-1)(x - 3)$

Now write this equation in standard quadratic form $ax^2 + bx + c = 0$ by using the Distributive Property and the two Properties of Equations from Page 1. This yields:

Eq. 4: $x^2 + 5x - 24 = 0$

This equation can be solved by using factoring and The Zero Factor Property

Eq. 5: $(x + 8)(x - 3) = 0$

and by The Zero Factor Property Eq. 5 is equivalent to:

Eqs. 6: $x + 8 = 0$ OR $x - 3 = 0$

The union of the solution sets for Eqs. 6 is $\{-8\}\cup\{3\}=\{-8,3\}$.

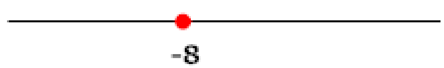
This is a good point to pause and review how the above equations are related.
Eq. 3, Eq. 4, Eq. 5, and Eqs. 6 are equivalent and their common solution set is $\{-8, 3\}$;
Eq. 1 and Eq. 2 are equivalent and their solution set is an unknown subset of $\{-8, 3\}$;
Eq. 2 and Eq. 3 are probably not equivalent.

It is critical to realize that the solution set for Eq. 1 is a subset of $\{-8, 3\}$.
Therefore -8 and 3 are POSSIBLE solutions for Eq. 1 and just as importantly they are the ONLY possibilities. This means we can now determine the solution set for Eq. 1 by testing -8 and 3 in either Eq. 1 or Eq. 2. Why does testing in either one work?

Important Fact: For rational equations, the only check that is required is to determine if the possible solution causes a zero in any denominator of the original equation.

For this particular Eq. 1, −8 does not cause a denominator to be zero. Consequently −8 is a solution. However, 3 does cause a denominator in Eq. 1 to be 0. Therefore 3 is not a solution. Since −8 and 3 are the only possible solutions for Eq. 1, we conclude the solution set for Eq. 1. is {−8}.

Because the equation is an equation in one variable, its graph is on the Real Number Line. The graph consists of a single dot at the number −8 on the number line.



For rational equations in one variable, the test of all possible solutions is an important and integral part of the solution process. This is significantly different than for linear equations in one variable where the test is not required. This significant difference is caused by the fact that when both sides of an equation are multiplied by an expression containing a variable, the resulting equation need not be equivalent to the original equation.