Operating With Equations – Another Attempt

There are certain fundamental elementary facts about equations that are absolutely necessary if you expect to solve equations correctly. Fortunately these are elementary facts and are easy to understand.

Definition: An equation is a mathematical statement which contains an = symbol.

Definition: A solution of an equation is a number (or numbers) which makes the equation true when substituted for the variable (or variables).

Definition: The solution set of an equation is the set of all solutions for that equation.

Discussion: To solve an equation means to find the solution set for the equation.

To solve an equation does not mean finding some of the solutions of the equation.

To solve an equation does not mean "find x". In any equation x may be any number for which the expressions makes sense. Some of the values for x make the equation true and some of the values of x make the equation false.

The process of solving an equation is any legitimate mathematical process which yields all numbers which make the equation true.

Definition: Two equations are equivalent if they have the same solution sets.

Discussion: If the solution set for one equation is A and the solution set for another equation is B, then the two equations are equivalent if and only if $A = B$. Notice that the equations are equivalent – they are not equal. Equations are never equal. It is meaningless to claim that two equations are equal. Notice that the solution sets are equal – they are not equivalent. Sets are never equivalent. It is meaningless to claim that two sets are equivalent.

Property 1: If the same expression is added to both sides of an equation the resulting equation is equivalent to the original equation.

Discussion: Any number, including zero, as well as any variable expression may be added to both sides of an equation without changing the solution set. If any number is added to both sides of an equation, the resulting equation is equivalent to the original equation. If an expression containing a variable is added to both sides of an equation, the resulting equation is equivalent to the original equation.

Property 2: If both sides of an equation are multiplied by the same non-zero real number the resulting equation will be equivalent to the original equation.

Discussion: Both sides of an equation may be multiplied by a non-zero real number without changing its solution set. Multiplication by something other than a non-zero real number will probably change the solution set. Multiplication by an expression containing a variable will probably change the solution set.

Discussion: The above two properties (Property 1 and Property 2) are the only two operations that may be performed to or on both sides of an equation without the possibility of changing the solution set of the equation. If any other operation such as squaring, taking square roots, or multiplying both sides by a variable expression is performed to both sides of an equation, the resulting equation will probably not be equivalent to the original equation.

Common Error: The following erroneous statement is frequently heard in conversation or seen on papers. "If you do the same thing to both sides of an equation the resulting equation will be equivalent to the original."
That is **ABSOLUTE NONSENSE!** Consider this counterexample:
The solution set for the simplest equation $x = 3$ is clearly $\{3\}$
Multiply both sides by x to obtain the equation
$x^2 = 3x$ whose solution set is $\{0, 3\}$.
Clearly the two equations are NOT equivalent.

Process 1: Using Property 1 and Property 2

Property 1 and Property 2 above may be used with any equation in any part of mathematics. Although these properties are usually introduced during the study of solving linear equations, they are valid for all equations.

One reason these two properties are introduced during a discussion of solving linear equations is that every linear equation may be solved using only these two properties. It can be verified that for any linear equation Property 1 and Property 2 may be used to generate a sequence of equations, each equation equivalent to the previous equation, which terminates in a simplest equation. This simplest equation is therefore equivalent to the original equation. The solution set for this simplest equation is obvious and is also the solution set for the original equation.

These two properties, Property 1 and Property 2, constitute a complete discussion about the process of solving linear equations. What that means is that Property 1 and Property 2 tell you everything there is to know about solving linear equations. It also tells you that linear equations will not be mentioned in the following sections.

When the equation to be solved is not a linear equation, Property 1 and Property 2 are still valid and are frequently used. However, for a non-linear equation, Property 1 and Property 2 are not the only tools required to find the solution set.

In some cases these other tools produce equations which are not equivalent to the original equation. That fact complicates the process of solving the equation. Some of these complications are discussed in the following sections.

Process 2: Using the Quadratic Formula

The statement of the Quadratic Formula is:
All solutions of the equation $ax^2 + bx + c = 0$ where a, b, and c are Real Numbers with $a \neq 0$ are given by the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Notice this formula yields two simplest equations:
$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and the desired solution set is the union of the two solution sets of the simplest equations. Therefore the solution set of the quadratic equation is:

$$\left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

Since every quadratic equation in one variable may be written in the form specified in the Quadratic Formula, no other process is needed to solve quadratic equations. However, sometimes the process of factoring together with the Zero Factor Property is faster and simpler.

Zero Factor Property: If a and b are real numbers and ab =0, then a = 0 or b = 0.

Discussion: The Zero Factor Property is a formal statement of the familiar fact that if a product of two real numbers is zero, then one or both of the factors is zero. This property may be extended to any number of factors. So if a product of any number of real numbers is zero then one or more of the factors is zero.

Since the term factoring means to write an expression as a product, it is not surprising that factoring and the Zero Factor Property are “natural” partners for solving equations as described in the next section.

Process 3: Using Factoring and the Zero Factor Property

If an equation is of the form $ax^2 + bx + c = 0$ and the second degree polynomial can be factored into the product of two linear factors with real coefficients such that $ax^2 + bx + c = (dx + e)(fx + g)$, then the original equation may be rewritten as $(dx + e)(fx + g) = 0$. According to the Zero Factor Property then $dx + e = 0$ or $fx + g = 0$. Each of these two linear equations is easily solved using Property 1 and Property 2.

The important consideration is that the original equation $ax^2 + bx + c = 0$ is equivalent to the two equations $dx + e = 0$ or $fx + g = 0$ joined with the conjunction OR. What this means is that the solution set of the original equation is equal to the union of the solution sets for the two linear equations.

This may be extended to more general polynomial equations in the following way. If an equation has the form "polynomial = 0" and if the polynomial can be factored into linear and quadratic factors (for this discussion call them L1, L2, Q1, and Q2), then the solution set for the original equation is equal to the union of the solution sets for the equations L1 = 0, L2 = 0, Q1 = 0, and Q2 = 0. The reader should recognize that this works for any number of factors which might be involved.

Process 4: Squaring both sides of an equation

In general, when both sides of an equation are squared, the resulting equation will not be equivalent to the original equation. Therefore if both sides of an equation are squared it must be assumed that the resulting equation is not equivalent to the original. This observation seems to indicate that squaring both sides of an equation is not a useful tool. However, there is a remedy.

When both sides of an equation are squared the solution set of the original equation is a subset of the solution set of the resulting equation.

This means that all the solutions of the original equation are contained in the solution set of the new equation. All that is needed, is to check each solution of the new equation in the original equation. Some will be solutions and some will not be solutions. Note that in this situation checking is an important required part of the solving process. Checking is not a required part of the solving process in any of the previously discussed processes.

Process 5: Multiplying both sides of an equation by a variable

In general, when both sides of an equation are multiplied by an expression containing a variable, the resulting equation will not be equivalent to the original equation. Therefore if both sides of an equation are multiplied by an expression containing a variable it must be assumed that the resulting equation is not equivalent to the original. This observation seems to indicate that multiplying both sides of an equation by an expression containing a variable is not a useful tool. However, just as when squaring both sides of an equation, there is a remedy to this dilemma.

When both sides of an equation are multiplied by an expression containing a variable the solution set of the original equation is a subset of the solution set of the resulting equation.

This means that all the solutions of the original equation are contained in the solution set of the new equation. All that is needed, is to check each solution of the new equation in the original equation. Some will be solutions and some will not be solutions. Note that in this situation checking is an important required part of the solving process.

When solving so called rational equations, equations containing rational expressions, it is frequently desirable to multiply both sides of the equation by a common denominator. When some of the denominators contain variables, both sides of the equation will be multiplied by an expression containing a variable and therefore a check must be part of the solving process. However, if the multiplier consists only of denominators, the check may be simplified. In this case it is sufficient to determine whether or not a potential solution causes a zero in a denominator of the original equation.

Process 6: Solving Equations Involving Absolute Value

Definition: The absolute value of a number is its distance from 0 on the number line. A more precise algebraic definition is:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The basis for solving all equations involving absolute values is the definition of absolute value.
Notice the precise definition of absolute value has two cases:

Case 1: The expression inside the absolute value symbol is positive or zero.

Case 2: The expression inside the absolute value symbol is negative.

Every equation involving absolute value is solved by considering these two cases as demonstrated in the following general example:

To solve an equation involving |something|, two cases must be considered.
The two cases arise from the definition of absolute value.
Therefore to solve any equation involving |something|, we consider:

Case 1: The equation that results from replacing |something| with (something)

Case 2: The equation that results from replacing |something| with -(something).