

## Examples of Functions

In this document is provided examples of a variety of functions. The purpose is to convince the beginning student that functions are something quite different than polynomial equations.

These examples illustrate that the domain of functions need not be the real numbers or any other particular set. These examples illustrate that the rule for a function need not be an equation and may in fact be presented in a great variety of ways.

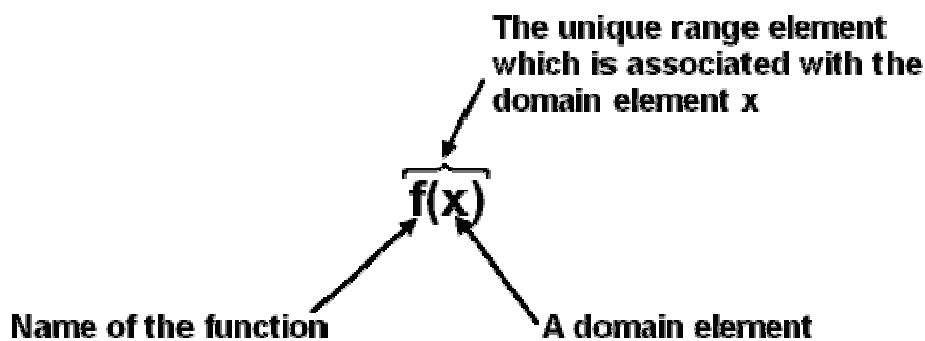
Some of the functions receive more complete treatment/discussion than others. If a particular function or function type is normally a part of a College Algebra course, then a more complete discussion of that function and its properties is likely.

Throughout this document the following definition of function, functional notation and convention regarding domains and ranges will be used.

**Definition:** A function consists of three things;

- i) A set called the domain
- ii) A set called the range
- iii) A rule which associates each element of the domain with a unique element of the range.

**Functional Notation:**



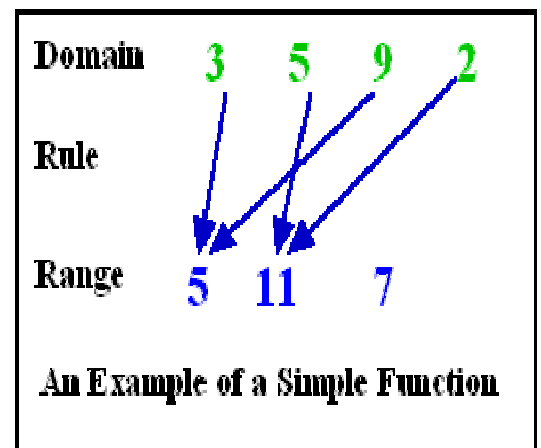
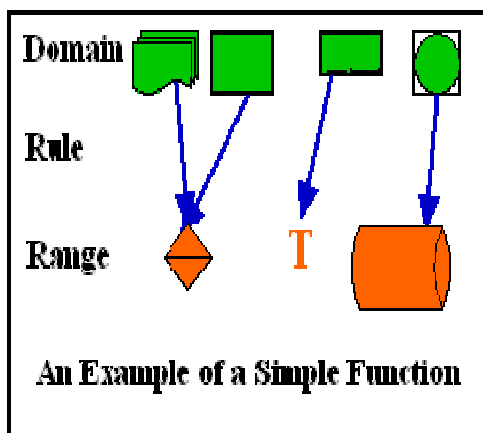
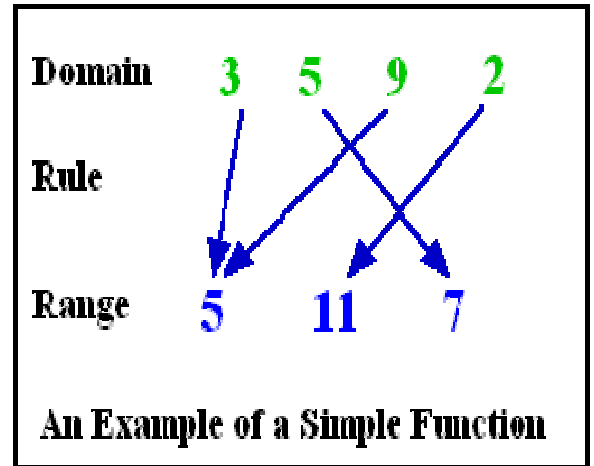
**Convention:** When the domain or range of a function is not specified, it is assumed that the range is  $\mathbf{R}$ , and the domain is the largest subset of  $\mathbf{R}$  for which the rule makes sense.

## I) Some Very Simple Functions

The domains and rule of the three functions shown in these diagrams are quite simple, but they are indeed functions.

Moreover view of functions consisting of two sets with arrows from elements of the domain to elements of the range will serve well to help understand functions in general

The example below shows quite clearly that the domain and range of a function need not be sets of numbers. In most College Algebra courses the functions discussed do have domains and ranges which are sets of numbers, but that is not a requirement of the definition of function.



## II) Linear Functions

**Definition:** A linear function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule can be expressed as a linear equation.

Recall a linear equation in two variables is an equation which can be written as  $y = mx + b$  where  $m$  and  $b$  are real numbers.

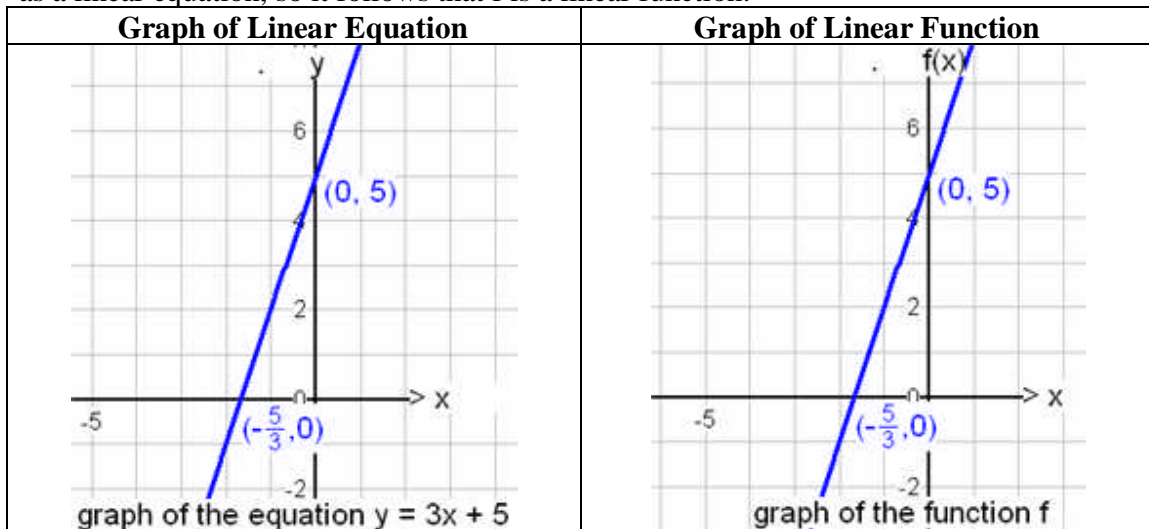
This leads to the following alternate **preferred** definition for a linear function.

**Definition:** A linear function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule can be written as  $f(x) = mx + b$  where  $x$  is a domain element,  $m \in \mathbf{R}$  and  $b \in \mathbf{R}$ .

**Comment:** Linear functions are a special case of the more general class of functions called polynomial functions. Thus the set of linear functions is a subset of the set of polynomial functions.

**Example:** Let  $f$  be the function whose rule is  $f(x) = 3x + 5$ .

Convention dictates that the range of this function is  $\mathbf{R}$  and because the rule makes sense for every real number substituted into the rule, the domain is also  $\mathbf{R}$ . The rule is written as a linear equation, so it follows that  $f$  is a linear function.



**Comment:** The word linear is now being used as an important adjective in four distinct contexts. We speak of linear polynomials, linear equations in one variable, linear equations in two variables, and linear functions.

- A linear polynomial is an expression which can be written as  $ax + b$  where  $a$  and  $b$  are real numbers. A linear polynomial is a mathematical creature just like natural numbers, rational numbers, irrational numbers, matrices, and vectors are mathematical creatures. There is no equation involved. The concept of solving a linear polynomial is meaningless. The arithmetic-like concepts of addition, multiplication, and subtraction in the context of all polynomials as well as linear polynomials does have meaning.
- A linear equation in one variable is an equation which can be written as  $ax + b = 0$  where  $a$  and  $b$  are real numbers. The graph of a linear equation in one variable is a single point on the  $x$ -axis unless  $a = 0$  and  $b \neq 0$  in which case there are no solutions and hence no graph.
- A linear equation in two variables is an equation which can be written in the form  $y = mx + b$  where  $m$  and  $b$  are real numbers. This is called the slope-intercept form of the equation of a line. The graph of a linear equation in two variables is a line. The coefficient  $m$  is the slope of the graph and  $b$  is the  $y$ -intercept of that graph.
- A linear function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule may be written as  $f(x) = mx + b$  where  $a$  and  $b$  are real numbers. The graph of a linear function is a non-vertical line with slope  $m$  and  $y$ -intercept  $b$ . A linear

function is a mathematical creature just like natural numbers, rational numbers, irrational numbers, matrices, vectors, and linear polynomials are mathematical creatures. The concept of solving a linear function is meaningless. The arithmetic-like concepts of addition, multiplication, and subtraction in the context of all functions as well as linear functions does have meaning. A variety of additional facts peculiar to functions will be studied in this course.

### III) Quadratic Functions

**Definition:** A quadratic function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule can be expressed as a quadratic equation.

Recall that a quadratic equation in two variables is an equation that can be written  $y = ax^2 + bx + c$  where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ .

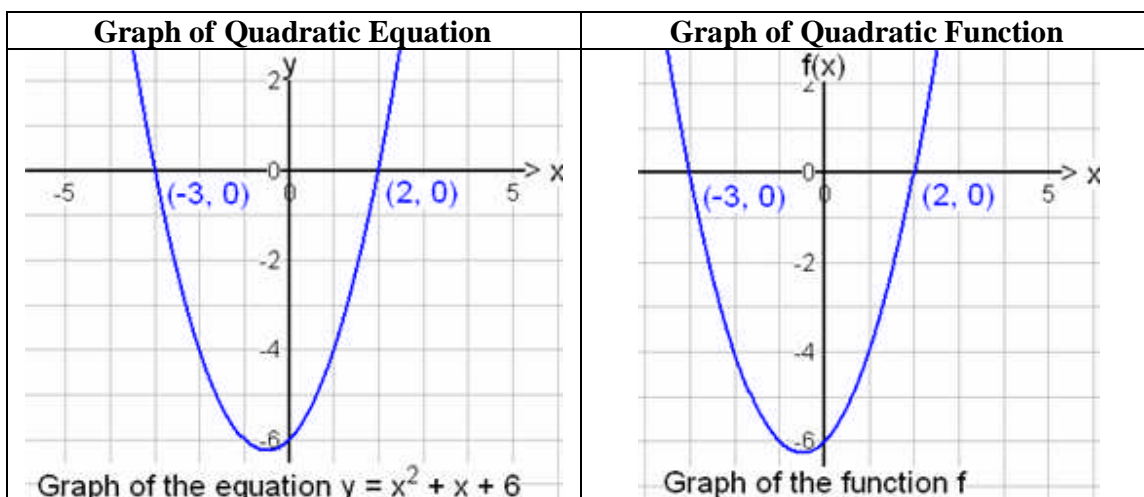
This leads to the following alternate **preferred** definition for a quadratic function.

**Definition:** A quadratic function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule can be written as  $f(x) = ax^2 + bx + c$  where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ .

**Comment:** Quadratic functions are a special case of the more general class of functions called polynomial functions. Thus the set of quadratic functions is a subset of the set of polynomial functions.

**Example:** Let  $f$  be the function whose rule is  $f(x) = x^2 + x - 6$ .

The convention dictates that the range of this function is  $\mathbf{R}$  and because the rule makes sense for every real number substituted into the rule, the domain is also  $\mathbf{R}$ . The rule is written as a quadratic equation, so it follows that  $f$  is a quadratic function.



**Comment:** The word quadratic is now being used as an important adjective in four distinct contexts. We speak of quadratic polynomials, quadratic equations in one variable, quadratic equations in two variables, and quadratic functions.

- A quadratic polynomial is an expression which can be written as  $ax^2 + bx + c$  where  $a$  and  $b$  are real numbers. A quadratic polynomial is a mathematical creature just like natural numbers, rational numbers, irrational numbers, matrices, vectors, linear polynomials, and linear functions are mathematical creatures. There is no equation involved. The concept of solving a quadratic polynomial is meaningless. The arithmetic-like concepts of addition, multiplication, and subtraction in the context of all polynomials as well as quadratic polynomials do have meaning.
- A quadratic equation in one variable is an equation which can be written as  $ax^2 + bx + c = 0$  where  $a$ ,  $b$ , and  $c$  are real numbers. The graph of a quadratic equation in one variable may be a single point, or two points on the  $x$ -axis. If the discriminant of the quadratic polynomial is negative the quadratic equation in one variable has no real solutions and hence has no graph.
- A quadratic equation in two variables is an equation which can be written in the form  $y = ax^2 + bx + c$  where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ . The graph of a quadratic equation in two variables is a parabola which opens up if the leading coefficient is positive and opens down if the leading coefficient is negative.
- A quadratic function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule may be written as  $f(x) = ax^2 + bx + c$  where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ . The graph of a quadratic function is a parabola which opens up if  $a > 0$  and opens down if  $a < 0$ . A quadratic function is a mathematical creature just like natural numbers, rational numbers, irrational numbers, matrices, vectors, and linear polynomials are mathematical creatures. The concept of solving a linear function is meaningless. The arithmetic-like concepts of addition, multiplication, and subtraction in the context of all functions as well as quadratic functions do have meaning. A variety of additional facts peculiar to functions will be studied in this course.

#### IV) Polynomial Functions

**Definition:** A polynomial function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule can be expressed as a polynomial equation.

Recall that a polynomial equation in two variables is an equation that can be written  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  where  $n$  is a whole number and each of the coefficients  $a_i$  is a real number.

This leads to the following alternate **preferred** definition for a quadratic function.

**Definition:** A polynomial function is a function whose domain is  $\mathbf{R}$ , whose range is  $\mathbf{R}$ , and whose rule can be expressed as  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  where  $n$  is a whole number and each of the coefficients  $a_i$  is a real number.

**Examples:** Every linear function is a polynomial function.

**Examples:** Every quadratic function is a polynomial function.

**Example:** Let  $f$  be the function whose rule is  $f(x) = x^5 + 3x^2 - 6x + 9$ .

Convention dictates that the range of this function is  $\mathbf{R}$  and because the rule makes sense for every real number substituted into the rule, the domain is also  $\mathbf{R}$ . The rule is written as a polynomial equation, so it follows that  $f$  is a polynomial function.

Polynomial functions will be a major topic of study in College Algebra.

## V) Absolute Value Function

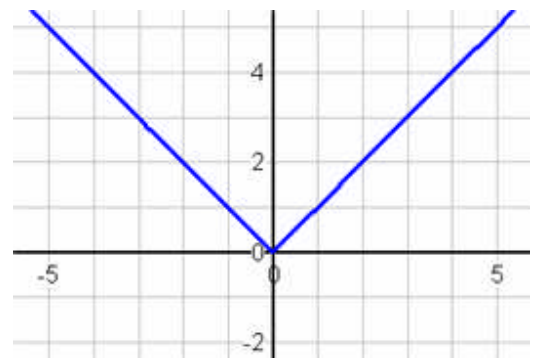
**Definition:** The absolute value function  $abs$  is the function whose rule is given by

$$abs(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Convention dictates that the range of this function is  $\mathbf{R}$  and because the rule makes sense for every real number substituted into the rule, the domain is also  $\mathbf{R}$ .

The graph of  $abs$  is shown at the right.

The name  $abs$  is less used in discussions involving the absolute value function. The more familiar notation  $|x|$  is usually used in lieu of  $abs(x)$ .



## VI) Sequences

**Definition:** A sequence is a function whose domain is the set of Natural Numbers  $\mathbf{N}$ .

The definition of sequence is pretty simply stated, but there are many special consequences of the fact that the domain of a sequence is  $\mathbf{N}$ . Sequences have been studied for centuries. Sequences have been studied with and without the concept of function. Quite a bit of special (mostly historical in origin) terminology and notation is used in a discussion of sequences.

When working with sequences, range elements are frequently called terms of the sequence. For example:

- The range element associated with the domain element 1 is called the first term of the sequence.

- The range element associated with the domain element 6 is called the sixth term of the sequence.
- The range element associated with the domain element  $n$  is called the  $n^{\text{th}}$  term of the sequence.

Because the domain of every sequence is  $\mathbf{N}$ , the graph of a sequence will consist of a set of discrete points.

Because the domain of every sequence is  $\mathbf{N}$ , it is possible to speak of the first domain element.

Because the domain of every sequence is  $\mathbf{N}$ , it is possible to speak of the next domain element.

Because the domain of every sequence is  $\mathbf{N}$ , it is not possible to speak of the last domain element.

Because the domain of every sequence is  $\mathbf{N}$ , it is possible to ask about and compute the sum of the first  $k$  terms. This is usually called the  $k^{\text{th}}$  partial sum of the sequence.

**Definition:** The  $n^{\text{th}}$  partial sum of a sequence is defined to be the sum of the first  $n$  terms of the sequence.

## VI - A) Tau

The function whose name is the Greek letter  $\tau$  (pronounced tau) is a function whose domain is the Natural Numbers  $\mathbf{N}$ . So  $\tau$  is a sequence. The rule for  $\tau$  is not given by a formula. The rule for  $\tau$  is:  $\tau(n)$  is the number of positive divisors of  $n$ .

To compute the range value associated with a particular domain element  $n$ , it is necessary to determine all positive divisors of  $n$  and simply count them. It is convenient to think of  $\tau$  simply as a function which counts the number of positive divisors of domain elements.

$$\begin{array}{cccccc} \tau(1) = 1 & \tau(2) = 2 & \tau(3) = 2 & \tau(4) = 3 & \tau(5) = 2 & \tau(6) = 4 \\ \tau(7) = 2 & \tau(8) = 4 & \tau(9) = 3 & \tau(10) = 4 & \tau(11) = 2 & \tau(12) = 6 \end{array}$$

The graph of the first 12 terms of Tau consists of the points:

$$\begin{array}{cccccc} (1, 1) & (2, 2) & (3, 2) & (4, 3) & (5, 2) & (6, 4) \\ (7, 2) & (8, 4) & (9, 3) & (10, 4) & (11, 2) & (12, 6) \end{array}$$

Notice that if  $p$  is a prime number, then  $\tau(p) = 2$  and if  $k$  is a composite number, then  $\tau(k) > 2$ . In fact sometimes  $\tau$  is used to define prime numbers as:

**Definition:** A natural number  $p$  is prime if and only if  $\tau(p) = 2$ .

Notice how neatly this definition prohibits classifying 1 as a prime number.

## VI - B) Sigma

The function whose name is the Greek letter  $\sigma$  (pronounced sigma) is a function whose domain is the Natural Numbers  $\mathbf{N}$ . So  $\sigma$  is a sequence. The rule for  $\sigma$  is not given by a formula. The rule for  $\sigma$  is:  $\sigma(n)$  is the sum of the positive divisors of  $n$ . To compute the range value associated with a particular domain element  $n$ , it is necessary to determine all positive divisors of  $n$  and simply add them.

$$\begin{array}{l} \sigma(1) = 1 \quad \sigma(2) = 3 \quad \sigma(3) = 4 \quad \sigma(4) = 7 \quad \sigma(5) = 6 \quad \sigma(6) = 12 \\ \sigma(7) = 8 \quad \sigma(8) = 15 \quad \sigma(9) = 13 \quad \sigma(10) = 18 \quad \sigma(11) = 12 \quad \sigma(12) = 28 \end{array}$$

The graph of the first 12 terms of Sigma consists of the points:

$$\begin{array}{cccccc} (1, 1) & (2, 3) & (3, 4) & (4, 7) & (5, 6) & (6, 12) \\ (7, 8) & (8, 15) & (9, 13) & (10, 18) & (11, 12) & (12, 28) \end{array}$$

## VI - C) Fibonacci Sequence

**Definition:** The Fibonacci sequence  $F$  is the function whose domain is  $\mathbf{N}$  and whose rule is given recursively by:  $F(1) = 1$ ,  $F(2) = 1$ , and for  $n > 2$ ,  $F(n) = F(n - 1) + F(n - 2)$

$$\begin{array}{l} F(1) = 1 \quad F(2) = 1 \quad F(3) = 2 \quad F(4) = 3 \quad F(5) = 5 \quad F(6) = 8 \\ F(7) = 13 \quad F(8) = 21 \quad F(9) = 34 \quad F(10) = 55 \quad F(11) = 89 \quad F(12) = 144 \end{array}$$

The graph of the first 12 terms of the Fibonacci sequence consists of the points:

$$\begin{array}{cccccc} (1, 1) & (2, 1) & (3, 2) & (4, 3) & (5, 5) & (6, 8) \\ (7, 13) & (8, 21) & (9, 34) & (10, 55) & (11, 89) & (12, 144) \end{array}$$

## VI - D) Arithmetic Sequences

**Definition:** An arithmetic sequence is a sequence whose consecutive terms have a common difference.

**Equivalent Definition:** An arithmetic sequence  $f$  is a function whose rule may be expressed as a linear equation of the form  $f(n) = dn + b$  where  $d$  is the common difference and  $b$  is the difference  $f(1) - d$ .

**Comments:** Compare the equivalent definition of an arithmetic sequence with the definition of a linear function.

The domain of a linear function is  $\mathbf{R}$       The domain of an arithmetic sequence is  $\mathbf{N}$

The rule for a linear function is  $f(x) = mx + b$

The rule for an arithmetic sequence is  $f(x) = dx + b$

The number  $b$  in the rule for an arithmetic sequence is the range value associated with 0 (if there were such a range element) so it corresponds exactly to the y-intercept of the linear function.



The common difference  $d$  is nothing more than the slope as you move from one range element to the next. The slope of the line joining two terms  $(x, f(x))$  and  $(x + 1, f(x + 1))$  of the sequence is given by  $\frac{f(x+1) - f(x)}{(x+1) - x} = \frac{d}{1} = d$ .

A casual approach is to view an arithmetic sequence as a linear function with domain  $\mathbf{N}$ .

**Comment:** Given any two pieces of information about an arithmetic sequence it is possible to determine its rule. The next three problem types illustrate the point.

**Problem Type 1:** If you are given the common difference and the first term of the arithmetic sequence, then it is possible to write the rule for the function. This is comparable to the slope-intercept situation/problem when working with linear functions.

**Example:** Suppose an arithmetic sequence named  $h$  has a common difference 8 and the first term is -5. Find the rule for the function  $h$ .

**Solution:** Since the function is an arithmetic sequence its rule is of the form  $h(n) = dn + b$ . In our case the common difference  $d$  is 8, the rule for  $h$  has the form  $h(n) = 8n + b$ . Because the first term is -5,  $b = -5 - 8 = -13$  and the rule for the desired arithmetic sequence is given by  $h(n) = 8n - 13$ .

**Problem Type 2:** If you are given the common difference  $d$  and one term of an arithmetic sequence, then it is possible to write the rule for the function. This is comparable to the point-slope situation/problems when working with linear functions.

**Example:** Suppose an arithmetic sequence named  $h$  has a common difference 3 and the fifth term is 12. Find the rule for the function  $h$ .

**Solution:** Since the function is an arithmetic sequence its rule is of the form  $h(n) = dn + b$ . In our case the common difference  $d$  is 3, the rule for  $h$  has the form  $h(n) = 3n + b$ . Because the fifth term is 12,  $h(5) = 12$ , but according to the partially determined rule  $h(5) = 3(5) + b = 15 + b$ . These two representations for  $h(5)$  yield the equation  $12 = 15 + b$ . Clearly then  $b = -3$  and the rule for the desired arithmetic sequence is given by  $h(n) = 3n - 3$ .

**Problem Type 3:** If you are given two terms of an arithmetic sequence, then it is possible to write the rule for the function. This is comparable to the two point situation/problems when working with linear functions.

**Example:** Suppose the fourth term an arithmetic sequence named  $k$  is 10 and the seventh term is 28. Find the rule for the function  $h$ .

**Solution:** Since the function is an arithmetic sequence its rule is of the form  $h(n) = dn + b$ . The difference between the seventh and fourth terms is  $3d$  and is also equal to  $28 - 10 = 18$ . That means  $3d = 18$  and so the common difference  $d$  is 6. The rule for  $h$  has the form  $h(n) = 6n + b$ . Because the fourth term is 10,  $h(4) = 10$ , but according to the partially determined rule  $h(4) = 6(4) + b = 24 + b$ . These two representations for  $h(4)$  yield the equation  $10 = 24 + b$ . Clearly then  $b = -14$  and the rule for the desired arithmetic sequence is given by  $h(n) = 6n - 14$ .

**Comment:** The formula for the  $n^{\text{th}}$  partial sum of an arithmetic sequence named  $a$  is:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

## VI - D) Mod Functions

### The function named $\text{mod}_6$

The name of this function is  $\text{mod}_6$ . The domain of  $\text{mod}_6$  is the set of all natural numbers. The range of  $\text{mod}_6$  is the set  $\{0,1,2,3,4,5\}$ . Because the domain of  $\text{mod}_6$  is  $\mathbf{N}$ ,  $\text{mod}_6$  is a sequence. The rule for  $\text{mod}_6$  is given by:

$\text{mod}_6(n)$  is the remainder when  $n$  is divided by 6.

To see that  $\text{mod}_6$  is a function, we refer back to the “arrow” concept of function. The fact that every natural number may be divided by 6 insures that an arrow emanates from each element of the domain.

The division algorithm states that for any natural number  $n$  there is a unique quotient  $q$  and a **unique** remainder  $r$  such that  $r \in \{0,1,2,3,4,5\}$ .

The fact that  $r \in \{0,1,2,3,4,5\}$  insures that the arrows end in the range and the fact that the remainder is unique insures that only one arrow emanates for each domain element.

Therefore  $\text{mod}_6$  is a function.

Here are examples of range elements associated with some domain elements

$$\text{mod}_6(3) = 3 \qquad \text{mod}_6(8) = 2 \qquad \text{mod}_6(17) = 5 \qquad \text{mod}_6(424) = 4$$

### The function named $\text{mod}_{11}$

The name of this function is  $\text{mod}_{11}$ . The domain of  $\text{mod}_{11}$  is the set of all natural numbers. The range of  $\text{mod}_{11}$  is the set  $\{0,1,2,3,4,5,6,7,8,9,10\}$ . Because the domain of  $\text{mod}_{11}$  is  $\mathbf{N}$ ,  $\text{mod}_{11}$  is a sequence. The rule for  $\text{mod}_{11}$  is given by:

$\text{mod}_{11}(n)$  is the remainder when  $n$  is divided by 11.

To see that  $\text{mod}_{11}$  is a function, we refer back to the “arrow” concept of function. The fact that every natural number may be divided by 11 insures that an arrow emanates from each element of the domain.

The division algorithm states that for any natural number  $n$  there is a unique quotient  $q$  and a **unique** remainder  $r$  such that  $r \in \{0,1,2,3,4,5,6,7,8,9,10\}$ .

The fact that  $r \in \{0,1,2,3,4,5,6,7,8,9,10\}$  insures that the arrows end in the range and the fact that the remainder is unique insures that only one arrow emanates from each domain element.

Therefore  $\text{mod}_{11}$  is a function.

Here are examples of range elements associated with some domain elements

$$\text{mod}_{11}(3) = 3 \qquad \text{mod}_{11}(8) = 8 \qquad \text{mod}_{11}(17) = 6 \qquad \text{mod}_{11}(424) = 6$$

It should be clear that for each natural number  $k$  there is a corresponding  $\text{mod}_k$  function defined in the same manner as  $\text{mod}_6$  and  $\text{mod}_{11}$  above.

## VII) Exponential Function exp

$e$  is one of those special numbers in mathematics, like pi, that keeps showing up in all kinds of important places. Like pi,  $e$  is an irrational number. The value of  $e$  may be approximated by  $e \approx 2.7182818284$

This irrational number is the foundation of a very important pair of functions in mathematics. These two functions are  $\exp$  and  $\ln$  (that is the letter ell). The function  $\exp$  is quite easy to define.

The function  $\exp$  has domain  $\mathbf{R}$  and range  $\mathbf{R}$ . The rule for  $\exp$  is given by the exponential equation  $\exp(x) = e^x$ . The graph of  $\exp$  is shown here.

The function  $\exp$  is called the exponential function base  $e$ .

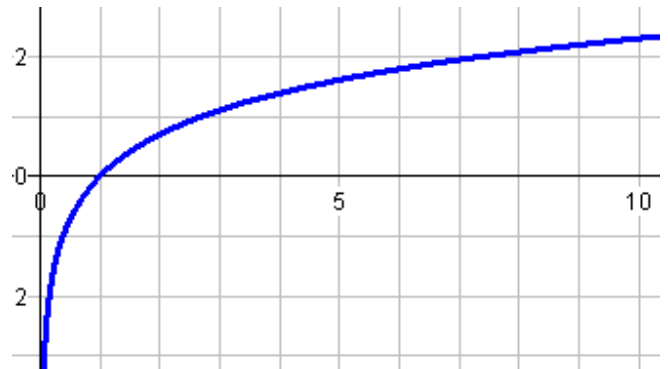
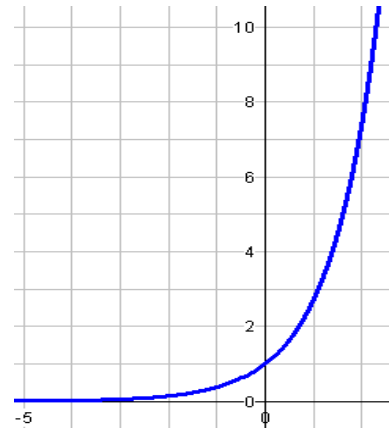
The function  $\ln$  is called the natural logarithm function.

The domain of  $\ln$  is all positive real numbers.

The rule for  $\ln$  is given in terms of its interrelation with the function  $\exp$ . The function  $\ln$  is the one and only function which has the property that

$$\ln(\exp(x)) = x \text{ and } \exp(\ln(x)) = x$$

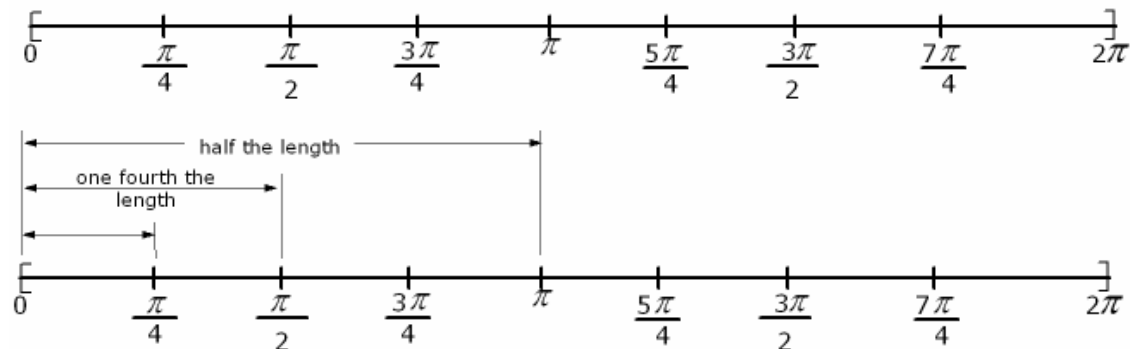
The graph of  $\ln$  is shown at the right.



## VIII) Circular Functions

We will now consider the functions whose names are sn, and cs, whose domain is the closed interval  $[0, 2\pi]$  and whose range is the closed interval  $[-1, 1]$ . These and four other functions are called circular functions or more traditionally Trigonometric functions.

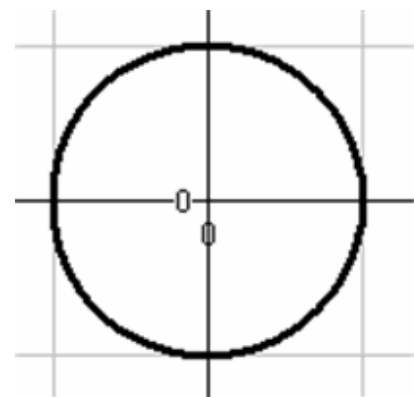
Observe the length of this domain is  $2\pi$ . Here is a picture of the domain of the function named sn. A few points which we will use later in the discussion are marked.



The rule for this function will not be described with an equation but will instead be described in terms of the coordinates of points on the unit circle.

Recall that the unit circle is the circle with radius 1 whose center is at the origin of the Cartesian coordinate system and is described by the equation  $x^2 + y^2 = 1$ .

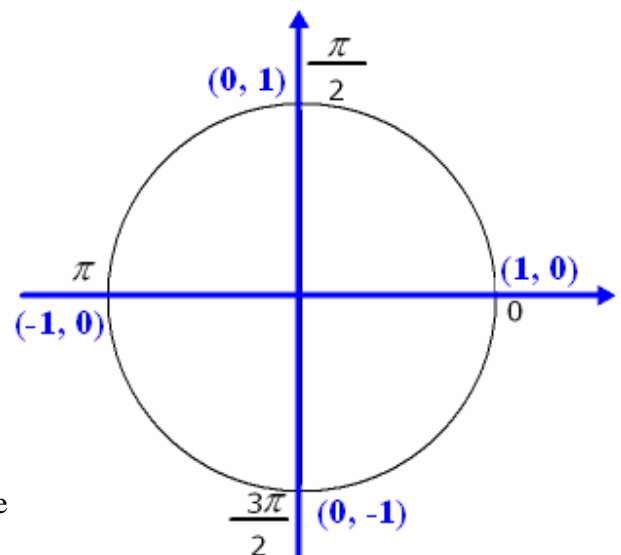
Also recall that the radius of a circle is given by the formula  $C = 2\pi r$ . In the case of the unit circle, the circumference is  $2\pi$ . This circumference is exactly the same length as the domain of the function named sn.



The significance of this comparison is that for any real number in the domain of the function named sn, there is a corresponding point on the unit circle. The converse is also true, for every point on the circumference of the unit circle there is a real number in the domain of the function.

On the unit circle the point  $(1, 0)$  is always considered the starting point and distance is always measured on the circumference in the counterclockwise direction.

- The point on the unit circle which corresponds to  $\pi/2$  in the domain of sn is the point with coordinates  $(0, 1)$ .



- The point on the unit circle which corresponds to  $\pi$  in the domain of sn is the point with coordinates  $(-1, 0)$ .
- The point on the unit circle which corresponds to  $3\pi/2$  in the domain of sn is the point with coordinates  $(-1, -1)$ .
- The point on the unit circle which corresponds to  $2\pi$  in the domain of sn is the point with coordinates  $(1, 0)$ .

Whether in the domain of sn or on the circumference of the unit circle, these four points are at the starting point,  $\frac{1}{4}$  the total distance,  $\frac{1}{2}$  the total distance, and  $\frac{3}{4}$  the total distance.

We are now ready to provide the rule for the functions named sn and cs.

**RULE:** For any  $x \in [0, 2\pi]$ , sn(x) is the second coordinate of the corresponding point on the circumference of the unit circle.

**RULE:** For any  $x \in [0, 2\pi]$ , cs(x) is the first coordinate of the corresponding point on the circumference of the unit circle.

The above diagram shows that:

$$\text{sn}(0) = 0, \quad \text{sn}\left(\frac{\pi}{2}\right) = 1, \quad \text{sn}(\pi) = 0, \quad \text{sn}\left(\frac{3\pi}{2}\right) = -1, \quad \text{sn}(2\pi) = 0$$

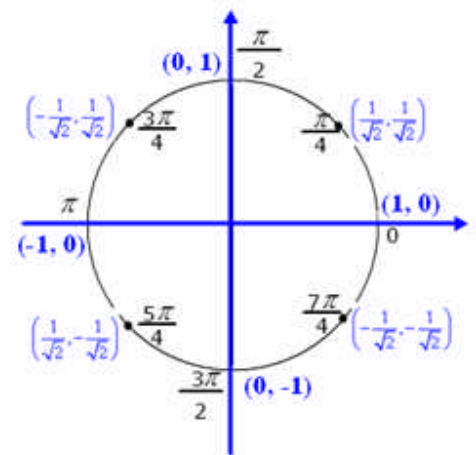
$$\text{cs}(0) = 1, \quad \text{cs}\left(\frac{\pi}{2}\right) = 0, \quad \text{cs}(\pi) = -1, \quad \text{cs}\left(\frac{3\pi}{2}\right) = 0, \quad \text{cs}(2\pi) = 1$$

We will now look at the range values associated with a few other domain elements. In particular we will examine those numbers (domain elements) midway between each pair of the previous four numbers in the domain of sn and cs.

Recall the rules for the functions sn and cs and extract the following range values directly from the picture at the right.

$$\text{sn}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \text{sn}\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \text{sn}\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad \text{sn}\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

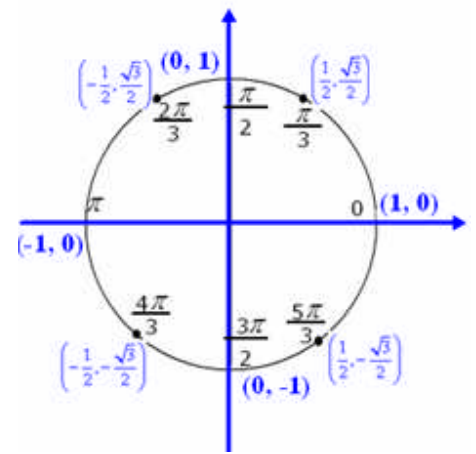
$$\text{cs}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \text{cs}\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad \text{cs}\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \text{cs}\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$



The figure at the right shows additional points in the domain of sn and cs with their coordinates on the unit circle. From this diagram and the rules for the two functions we can conclude:

$$\text{sn}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \text{sn}\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \text{sn}\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}, \quad \text{sn}\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\text{cs}\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad \text{cs}\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \quad \text{cs}\left(\frac{4\pi}{3}\right) = -\frac{1}{2}, \quad \text{cs}\left(\frac{5\pi}{3}\right) = \frac{1}{2}$$



In a manner similar to the examples presented in these few examples, the unique range values associated with a domain element may be determined.

Each point in the domain of sn and cs corresponds with a point on the unit circle which in turn corresponds with a set of first and second coordinates which determine the unique range value associated with the domain element.

The two functions sn and cs and four other circular functions are the functions studied in Trigonometry. How these functions relate to angles, triangles, radian measure, etc. will not be discussed here. The purpose here is simply to give an illustration of some functions whose rules are unusual and whose domains and ranges are not all of  $\mathbf{R}$ .