

Important Properties of Linear Equations in Two Variables

Definition: A **linear equation in two variables** is an equation which may be written in the form $y = mx + b$ where m , and b are real numbers.

The graph of a linear equation in two variables is a non-vertical line with slope m and y -intercept b . Every non-vertical line is the graph of a linear equation in two variables of the form $y = mx + b$

Because lines and linear equations in two variables are simple concepts, there are relatively few questions one can ask about them.

The only possible questions are:

When given a linear equation in two variables:

1. Write the equation in the form $y = mx + b$ or standard form.
2. Sketch the graph of the equation ?
3. What is the y -intercept of the graph of the equation ?
4. What is the x -intercept of the graph of the equation?
5. Is the graph increasing or decreasing ?

When asked to determine a linear equation in two variables:

6. Determine the linear equation in two variables whose graph has a given slope and a given y -intercept.
7. Determine the linear equation in two variables whose graph has a given slope and contains a given point..
8. Determine the linear equation in two variables whose graph contains two given points.

To answer these questions:

- 1) To obtain the slope-intercept form if the equation is not in the slope-intercept form, solve the equation for y . To obtain the standard form, add expressions to both sides to obtain $Ax + By = C$
- 2) **Process:** To sketch the graph of a linear equation in two variables, plot any two points whose coordinates satisfy the equation and draw the line passing through the two points. Plotting the two intercepts is generally a good idea.
- 3) The y -intercept occurs when $x = 0$. Therefore the y -intercept is b .
- 4) The x -intercept occurs when $y = 0$. Therefore we find the x -intercept by solving $mx + b = 0$ for x .
- 5) Whether the graph of a linear equation in two variables is increasing or decreasing depends on the leading coefficient (the slope m) when the equation is written in the slope-intercept form $y = mx + b$.

The graph is increasing if $m > 0$	The graph is decreasing if $m < 0$
The graph is increasing if the slope is positive	The graph is decreasing if the slope is negative
The graph is horizontal if $m = 0$	
The graph is horizontal if the slope is zero	

If two independent pieces of information are known about a line L it is possible to determine the linear equation in two variables whose graph is that line L . The most important fact used in this process is:

Definition: A point is on the graph of an equation if and only if its coordinates satisfy the equation.

That is, a point (t, k) is on the graph of an equation f if and only if substituting t and k into the equation yields a true statement.

How Does This Work?

Case 1: (Question 6) If the slope and y -intercept are known, replace m and b in $y = mx + b$ with the known values.

Case 2: (Question 7) If the slope and a point are known, replace m in $y = mx + b$ with the known value for m . Then use the fact that the point is on the graph to obtain an equation with b as the only unknown. Solve for b and substitute it into the partially determined equation.

Case 3: (Question 8) If two distinct points are known, calculate the slope of the line segment joining the two points, select one of the points and revert to Case 2.

Example of Case 2: Find the linear equation in two variables whose graph has slope 3 and passes through the point $(4, -3)$.

Solution: Since the desired equation is linear in two variables, it has the form $y = mx + b$.

Since its slope is 3, it has the form $y = 3x + b$. (*)

Since $(4, -3)$ is on the graph, its coordinates satisfy the equation.

That means a true statement results when 4 and -3 are substituted into (*)

Using (*) we get $-3 = (3)(4) + b$

Solving for b yields $b = -15$.

Substitute that value for b into the partially determined equation given in (*) to obtain

$y = 3x - 15$ as the linear equation in two variables whose graph is the line which passes through the point $(4, -3)$ with slope 3.

Alternate methods for determining the equation of a line depend on remembering formulas for each of the cases.

If the slope and y-intercept are known, use the
Slope Intercept Form of the Equation of a line
$$y = mx + b$$

If the slope and one point (x_1, y_1) on the line are known use the
Point-Slope Form of the equation of a line
$$y - y_1 = m(x - x_1)$$

If two points (x_1, y_1) and (x_2, y_2) are known to be on the line, then use the
Two Point Form of the equation of a Line

$$y - y_1 = \left(\frac{y_1 - y_2}{x_1 - x_2} \right) (x - x_1)$$

Standard Form for the equation of a line

$Ax + By = C$ where A, B, and C are real numbers and not both A and B are zero.

The **Equation of a Vertical Line** has the form $x = a$, where a is the x-intercept of the vertical line.

Definition: The **slope** of the non-vertical line through two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}$$

Fact: Two non-vertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. The statement that they are negative reciprocals of each other may be stated algebraically with any one of the following equations.

$$m_1 = \frac{-1}{m_2} \quad \text{or} \quad m_2 = \frac{-1}{m_1} \quad \text{or} \quad m_1 m_2 = -1$$

Fact: Two non-vertical lines are **parallel** if and only if they have different y-intercepts and they have the same slopes.

The following formulas are not related to linear equations, but are useful formulas to remember. To aid in the solution of certain word problems these formulas are frequently presented with the discussion of linear equations in two variables.

Fact: The **midpoint** of the line segment joining two points (x_1, y_1) and (x_2, y_2) is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Fact: The **distance** between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$