Law of Trichotomy - Linear Equations - Linear Inequalities in One and Two Variables

Law of Trichotomy: For any two real numbers a and b, exactly one of the following is true.

i. $a < b$
ii. $a = b$
iii. $a > b$

The Law of Trichotomy is a formal statement of a property which most of us would consider to be quite obvious; when comparing two numbers; they are equal, the first is less than the second, or the first is greater than the second. The purpose of the formal statement here is to call attention to the obvious fact and to make it available for use with algebraic quantities which represent real numbers.

Law of Trichotomy Applied to Linear Equations and Inequalities in One Variable

Example 1: During a consideration of the two linear algebraic expressions $3x + 5$ and $-2x + 7$, the Law of Trichotomy reminds us that there exists three distinct possibilities;

i. $3x + 5 < -2x + 7$
ii. $3x + 5 = -2x + 7$
iii. $3x + 5 > -2x + 7$

This means is that if x is replaced (in these three statements) by any individual real number, exactly one of the expressions will be true.
Another more precise way of stating this is that the union of the three solution sets is **R** (the real numbers) and the intersection of any two of the solution sets is the empty set.

Consequently we learn to solve the equation and both inequalities. Recall some of the observations made during the many practice exercises for solving linear equations and inequalities.

i. The graph of a linear equation in one variable is a point on the real number line.
ii. The graph of a linear inequality in one variable is a ray on the real number line.

What may not have been observed is that the ray which is the graph of a linear inequality in one variable begins at the graph of the equation and extends infinitely far toward the right or the left and that the graph of the other inequality begins at the same point and extends infinitely far in the other direction.

Another, possibly more understandable, way to state this is: The graph of a linear equation in one variable divides the real number line into two rays, one of which is the graph of one of the corresponding inequalities and the other is the graph of the other inequality.

Refer to Example 1. The solution set for the equation is $\left\{\frac{2}{5}\right\}$.

Fig.1

The graph of the equation 3x + 5 = –2x + 7 is shown in Fig. 1.

Clearly this graph of the equation divides the real number line into a point and two rays as shown in Fig. 2.The blue ray is the graph of one of the inequalities and the red ray is the graph of the other inequality. In this example the blue ray is the graph of 3x + 5 < –2x + 7 and the red ray is the graph of 3x + 5 > –2x + 7.

Fig. 2

We can determine if the blue ray is the solution set to one of the inequalities by testing just one number from the ray in the inequality. Refer to the above example. To determine if the blue ray is the solution to the inequality 3x + 5 > –2x + 7 we need only test one number from the blue ray in 3x + 5 > –2x + 7. The number 0 is in the blue ray and is easy to test. Substituting 0 into 3x + 5 > –2x + 7 yields 5 > 7 which is false. This allows a number of conclusions;

i. The blue ray is not the solution set for 3x + 5 > –2x + 7.
ii. The red ray (the other one) is the solution set for 3x + 5 > –2x + 7
iii. The blue ray is the solution set for 3x + 5 < –2x + 7.

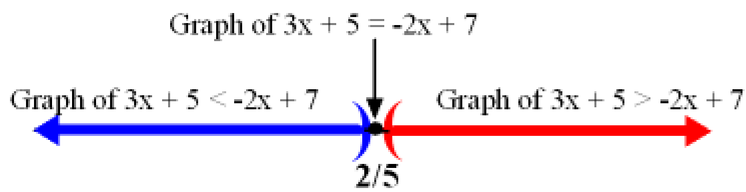


Fig. 3

The following observations can be generalized to many other situations. In particular they will apply to equations and inequalities involving nothing but polynomials.

i. When considering a conditional equation or inequality, the Law of Trichotomy dictates that we also consider the other two corresponding equations and/or inequalities.
ii. The graph of the equation is a boundary between the graphs of the corresponding inequalities. For that reason, the equation is sometimes called the boundary equation for the inequalities.
iii. Testing any single number from one of the rays in either inequality determines whether that ray is the solution set for that inequality.

Law of Trichotomy Applied to Linear Equations and Inequalities in Two Variables

A **linear equation in two variables** is an equation which may be written in the form Ax + By = C where A, B, and C are real numbers and B is not zero.

A **linear inequality in two variables** x and y is an inequality which can be written as Ax + By < C, Ax + By > C.

The Law of Trichotomy affects how we consider equations and inequalitites in two variables. When considering any one of the three possibilities Ax + By = C, Ax + By < C, or Ax + By > C, we consider all three.

The graph of a linear equation in two variables is a line in the coordinate plane which divides the coordinate plane into two halves. One of those half-planes is the graph of Ax + By < C, and the other half-plane is the graph of Ax + By > C.

When asked to analyze either of Ax + By = C, Ax + By < C, or Ax + By > C, we begin by graphing the equation by plotting two points (usually the x and y intercepts) and then testing one point (not on the boundary line) in one of the inequalities. Those few simple steps provide us with a complete analysis of the equation and both inequalities. The equation is called the **boundary equation** because its graph forms a boundary between the graphs of the two inequalities.

The following will illustrate the process and the results.
Example 2: Discuss the inequality 3x – 7y < 21.
Begin by analyzing the equation of the boundary line which is 3x – 7y = 21.
This is a linear equation in two variables so its graph is a line in the rectangular coordinate system.
If x = 0, then y = –3, so the point (0, –3) is the y-intercept of the boundary line.
If y = 0, then x = 7, so the point (7, 0) is the x-intercept of the boundary line.
Plot those two points and draw the line through them to obtain the graph of the boundary line as shown in green in Fig 20.

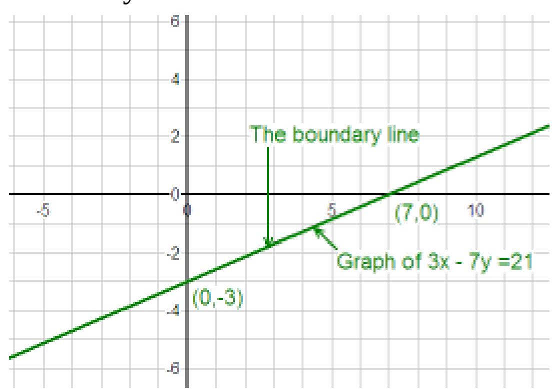
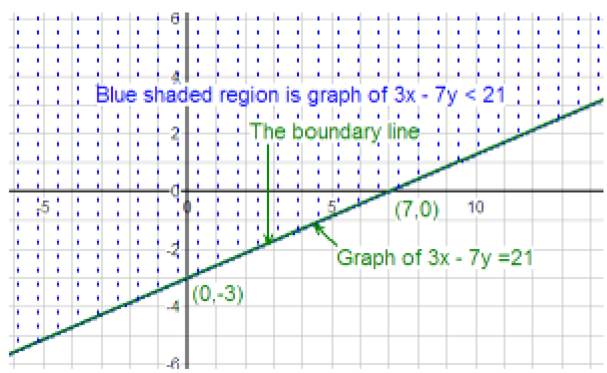


Fig 20

The origin is not on the graph of the boundary line and is easy to test in either of the inequalities.
Use (0, 0) as a test point in the inequality 3x – 7y < 21 to obtain
3(0) – 7(0) < 21 which is TRUE.
Therefore (0, 0) is a solution of 3x – 7y < 21 and every other point in that half-plane is a solution of the inequality 3x – 7y < 21.
Conclusion: The solution set for 3x – 7y < 21 is the half-plane containing the origin and bounded by the graph of 3x – 7y = 21.
We graph the inequality by shading the half-plane which is its solution set as shown in blue in Fig. 21.



Because of the Law of Trichotomy we know that the other half-plane (the un-shaded part) is the graph of 3x – 7y > 21.

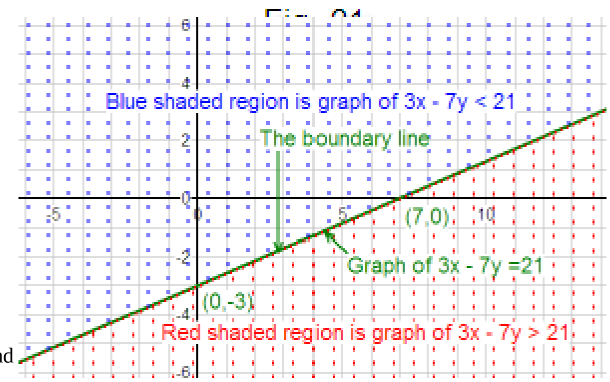


Fig. 22

Conclusion: The solution set for 3x – 7y > 21 is the half-plane NOT containing the origin and bounded by the graph of 3x – 7y = 21.
For consistency we show this graph in red in Fig. 22.

Example 3: Discuss the system of inequalities $\begin{cases} 2x + 5y \geq 10 \\ -3x + 2y \geq 6 \\ 2x - 7y > -38 \end{cases}$

Begin by observing that this is a system of three linear inequalities in two variables. The solution set for each individual inequality is a half-plane. (The Law of Trichotomy).
The solution set for the system is the set of points which are solutions of 2x + 5y ≥ 10 AND –3x + 2y ≥ 6 AND 2x – 7y > – 38. Therefore the solution set of the system is the intersection of the three individual solution sets.
A good strategy is to graph each of the individual inequalities in different colors on the same coordinate system so the intersections is easily observed to be the region shaded in all three colors. For instructional purposes, each inequality will initially be graphed on its own coordinate system and then they will be combined.

The boundary equation for 2x + 5y ≥ 10 is 2x + 5y = 10.
If x = 0, then y = 2. Therefore (0, 2) is the y-intercept of the boundary equation.
If y = 0, then x = 5. Therefore (5, 0) is the x-intercept of the boundary equation.
The origin is not on the boundary line so we test it in 2x + 5y ≥ 10 to obtain 0 ≥ 10 which is FALSE.
Therefore the graph of 2x + 5y ≥ 10 is the half-plane which does not contain the origin and is bounded by the boundary line 2x + 5y = 10. Because the symbol ≥ permits equality, the boundary line is part of the solution set and is therefore drawn as a solid line to indicate that it is part of the graph.
The graph of 2x + 5y ≥ 10 appears in Fig. 23.

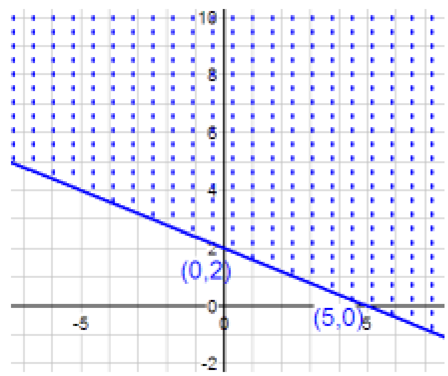


Fig. 23

The boundary equation for –3x + 2y ≥ 6 is –3x + 2y = 6.
If x = 0, then y = 3. Therefore (0, 3) is the y-intercept of the boundary equation.
If y = 0, then x = –2. Therefore (–2, 0) is the x-intercept of the boundary equation.
The origin is not on the boundary line so we test it in –3x + 2y ≥ 6 to obtain 0 ≥ 6 which is FALSE.

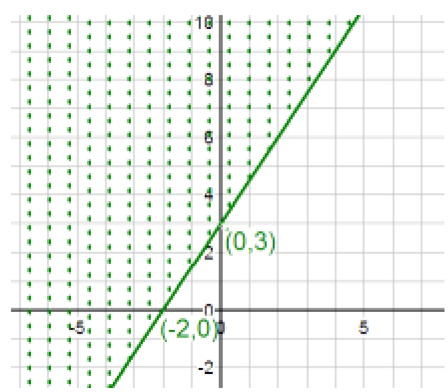


Fig. 24

Therefore the graph of $-3x + 2y \geq 6$ is the half-plane which does not contain the origin and is bounded by the boundary line $-3x + 2y = 6$. Because the symbol $\geq$ permits equality, the boundary line is part of the solution set and is therefore drawn as a solid line to indicate that it is part of the graph.
The graph of $-3x + 2y \geq 6$ appears in Fig. 24.

The boundary equation for $2x - 7y > -38$ is $2x - 7y = -38$.
If x = 0, then $y = \frac{38}{7}$. Therefore $\left(0, \frac{38}{7}\right)$ is the y-intercept of the boundary equation.
If y = 0, then x = –19. Therefore (–19, 0) is the x-intercept of the boundary equation.
The origin is not on the boundary line so we test it in $2x - 7y > -38$ to obtain $0 \geq -38$ which is TRUE.
Therefore the graph of $2x - 7y > -38$ is the half-plane which contains the origin and is bounded by the boundary line $2x - 7y = -38$. Because the symbol > does not permit equality, the boundary line is not part of the solution set and is therefore drawn as a dashed line to indicate that it is not part of the graph.
The graph of $2x - 7y > -38$ appears in Fig. 25.

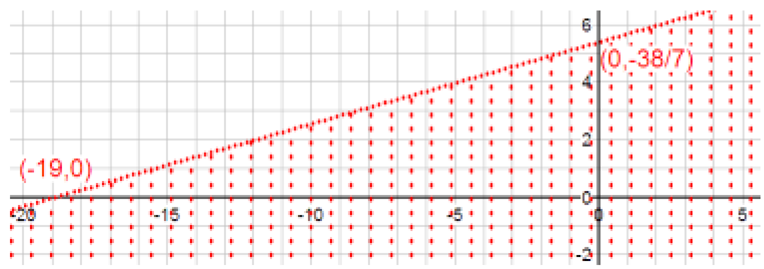


Fig. 25

Recall that the graph of the system is the intersection of the graphs of the individual inequalities in the system. In Fig. 26 the three graphs are superimposed on the same coordinate system and their intersection becomes clear.
The graph of the system of inequalities is shown in Fig. 26.
The solution of the system

$$\begin{cases} 2x + 5y \geq 10 \\ -3x + 2y \geq 6 \\ 2x - 7y > -38 \end{cases}$$

is the triangular region containing red, blue, and green shading and bounded by the three boundary lines. Note the blue and green boundaries of the triangle are part of the graph and the red boundary is not part of the graph.
This graph tells us that every point in the triangle or on the red or on the blue edge of the triangle is a solution of the system of inequalities.

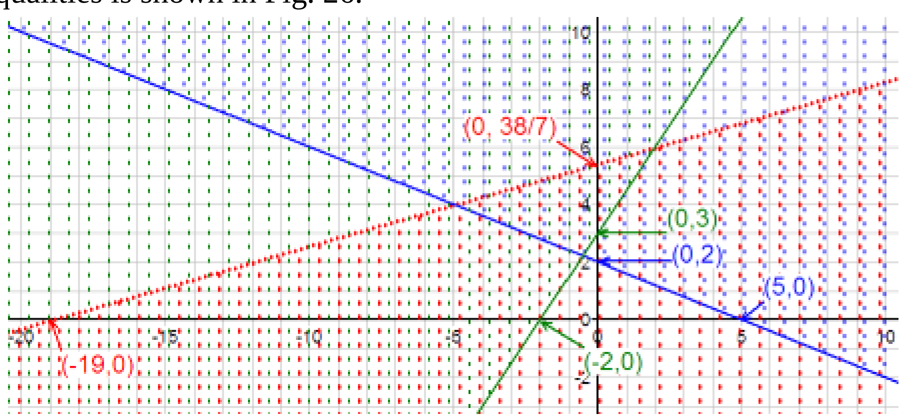


Fig. 26

In terms of sets and unions we see that the solution set of the system is the set of points in the interior of the triangle UNION the set of points on the blue boundary of the triangle UUNION the set of points on the green boundary of the triangle.