Law of Trichotomy and Boundary Equations

Law of Trichotomy: For any two real numbers a and b, exactly one of the following is true.

i. $a < b$
ii. $a = b$
iii. $a > b$

The Law of Trichotomy is a formal statement of a property which most of us would consider to be quite obvious; when comparing two numbers; they are equal, the first is less than the second, or the first is greater than the second. The purpose of the formal statement here is to call attention to the obvious fact and to make it available for use with algebraic quantities which represent real numbers.

Law of Trichotomy Applied to Linear Inequalities in One Variable:

Example 1: During a consideration of the two linear algebraic expressions 3x + 5 and –2x + 7, the Law of Trichotomy reminds us that there exists three distinct possibilities;

i. $3x + 5 < -2x + 7$
ii. $3x + 5 = -2x + 7$
iii. $3x + 5 > -2x + 7$

Consequently we learn to solve the equation and both inequalities. Recall some of the observations made during the many practice exercises for solving linear equations and inequalities.

i. The graph of a linear equation in one variable is a point on the real number line.
ii. The graph of a linear inequality in one variable is a ray on the real number line.

What may not have been observed is that the ray which is the graph of a linear inequality in one variable begins at the graph of the equation and extends infinitely far toward the right or the left and that the graph of the other inequality begins at the same point and extends infinitely far in the other direction.

Another, possibly more understandable, way to state this is: The graph of a linear equation in one variable divides the real number line into two rays, one of which is the graph of one of the corresponding inequalities and the other is the graph of the other inequality.

Refer to Example 1. The solution set for the equation is $\left\{\frac{2}{5}\right\}$.

Fig.1

The graph of the equation $3x + 5 = -2x + 7$ is shown in Fig. 1.

Clearly this graph of the equation divides the real number line into a point and two rays as shown in Fig. 2.The blue ray is the graph of one of the inequalities and the red ray is the graph of the other inequality. In this example the blue

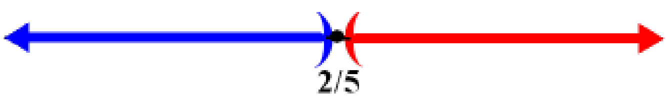

Fig. 2

ray is the graph of $3x + 5 < -2x + 7$ and the red ray is the graph of $3x + 5 > -2x + 7$. We can determine if the blue ray is the solution set to one of the inequalities by testing just one number from the ray in the inequality. Refer to the above example. To determine if the blue ray is the solution to the inequality $3x + 5 > -2x + 7$ we need only test one number from the blue ray in $3x + 5 > -2x + 7$. The number 0 is in the blue ray and is easy to test. Substituting 0 into $3x + 5 > -2x + 7$ yields $5 > 7$ which is false. This allows a number of conclusions;

i. The blue ray is not the solution set for $3x + 5 > -2x + 7$.
ii. The red ray (the other one) is the solution set for $3x + 5 > -2x + 7$
iii. The blue ray is the solution set for $3x + 5 < -2x + 7$.

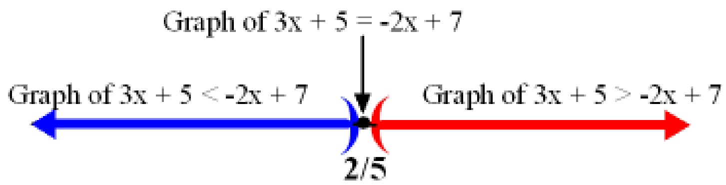

Fig. 3

The following observations can be generalized to many other situations. In particular they will apply to equations and inequalities involving nothing but polynomials.

i. When considering a conditional equation or inequality, the Law of Trichotomy dictates that we also consider the other two corresponding equations and/or inequalities.
ii. The graph of the equation is a boundary between the graphs of the corresponding inequalities. For that reason, the equation is sometimes called the boundary equation for the inequalities.
iii. Testing any single number from one of the rays in either inequality determines whether that ray is the solution set for that inequality.

Law of Trichotomy Applied to Quadratic Inequalities in One Variable:

Example 2: Suppose it is required to solve the inequality $x^2 + x - 6 < 0$.
Discussion and Solution:
When considering the inequality $x^2 + x - 6 < 0$ the Law of Trichotomy dictates that we be aware of the equality $x^2 + x - 6 = 0$ as well as the inequality $x^2 + x - 6 > 0$. As in the previous discussion, the equation $x^2 + x - 6 = 0$ is sometimes called the boundary equation because its graph is the boundary between the graphs of the two inequalities. Our strategy will be to graph the equation and test numbers from the various resulting rays and intervals formed by that graph.
Factoring and The Zero Factor Property show the solution set of the equation $x^2 + x - 6 = 0$ to be $\{2, -3\}$. We can now sketch the graph of the equation $x^2 + x - 6 = 0$ as shown in Fig. 4.

Fig. 4

The graph of the equation $x^2 + x - 6 = 0$ divides the real line into an interval $(-3, 2)$ (green) and two rays (blue $(-\infty, -3)$ and red $(2, \infty)$).

- The interval is part of the solution set for either $x^2 + x - 6 < 0$ or $x^2 + x - 6 > 0$.
- The blue ray $(-\infty, -3)$ is part of the solution set for $x^2 + x - 6 < 0$ or $x^2 + x - 6 > 0$.
- The red ray $(2, \infty)$ is part of the solution set for $x^2 + x - 6 < 0$ or $x^2 + x - 6 > 0$.

We need to test one number from each of the rays and the interval in either of the inequalities.

Test 0 from the interval (–3, 2) in the inequality $x^2 + x - 6 < 0$. When 0 is substituted into $x^2 + x - 6 < 0$ we obtain $-6 < 0$; a true statement. Therefore the interval is part of the solution set for $x^2 + x - 6 < 0$.
Test – 4 from the ray $(-\infty, -3)$ in the inequality $x^2 + x - 6 < 0$. When – 4 is substituted into $x^2 + x - 6 < 0$ we obtain $16 - 4 - 6 < 0$; a false statement. Therefore – 4, and consequently no number in the ray $(-\infty, -3)$, is a solution of the inequality $x^2 + x - 6 < 0$. It now follows from the Law of Trichotomy that every number in the ray $(-\infty, -3)$ is a solution of the other inequality $x^2 + x - 6 > 0$. Therefore the ray $(-\infty, -3)$ is part of the solution set for $x^2 + x - 6 > 0$.
Test 3 from the ray $(2, \infty)$ in the inequality $x^2 + x - 6 > 0$. (observe that I switched inequalities). When 3 is substituted into $x^2 + x - 6 > 0$ we obtain $9 + 3 - 6 >$ a true statement. Consequently the ray $(2, \infty)$ is part of the solution set for $x^2 + x - 6 > 0$.

A summary of the test results shows that the solution set for the inequality $x^2 + x - 6 > 0$ is $(-\infty, -3) \cup (2, \infty)$ and the solution set for $x^2 + x - 6 > 0$ is the interval $(-3, 2)$. Illustrated in Fig. 5.

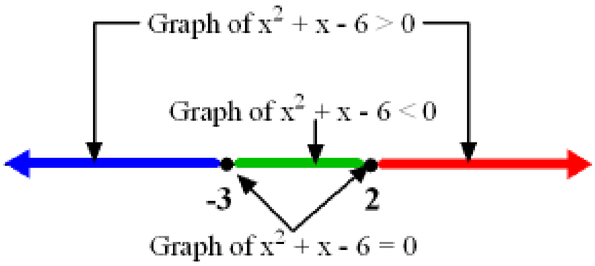

Fig. 5

Rule for Quadratics: When working with quadratic equations and inequalities in one variable;

- If the equation has two real solutions, the resulting interval will be the solution set for one of the inequalities and the union of the rays will be the solution set for the other inequality.
- If the equation has one real solution the union of the resulting two rays will be the solution set for one of the inequalities and the solution set for the other inequality is the empty set $\varnothing$.
- If the equation has no real solution, the solution set for one of the inequalities is the set of real numbers **R** and the solution set for the other inequality is the empty set $\varnothing$.

Law of Trichotomy Applied to Polynomial Inequalities in One Variable:

Rule: The above process may be used to solve any polynomial inequality in one variable.

This will be illustrated in Example 3.
Example 3: Solve the inequality x(x – 2)(x + 3)(x + 3)(x + 4) < 0.
Discussion and Solution:
Generally fifth degree polynomial inequalities in one variable will not be solvable, but when we can factor it, as in this case, we can solve the inequality
When considering the inequality x(x – 2)(x + 3)(x + 3)(x + 4) < 0 the Law of Trichotomy dictates that we be aware of the equality x(x – 2)(x + 3)(x + 3)(x + 4) = 0 as well as the inequality x(x – 2)(x + 3)(x + 3)(x + 4) > 0.The equation x(x – 2)(x + 3)(x + 3)(x + 4) = 0 is called the boundary equation because its graph is the boundary between the graphs of the two inequalities.
Our strategy will be to graph the equation and test numbers from the various resulting rays and intervals formed by that graph.
The Zero Factor Property shows the solution set of the equation x(x – 2)(x + 3)(x + 3)(x + 4) = 0 to be {– 4, –3, 0, 2}. A sketch of the graph of the equation x(x – 2)(x + 3)(x + 3)(x + 4) = 0 as shown in Fig. 6. Observe the rays and intervals are;

(– ∞, – 4), (–4, –3), (–3, 0), (0, 2), and (2, ∞)

Fig. 6

The solution set for the two inequalities can be determined by testing a number from each of the rays and intervals. However there is a simpler approach which depends on the fact that it is simple to determine the sign of a linear expression in an interval.
A convenient and effective way to organize the test is with a table as illustrated in Fig. 7.

	(-∞, -4)	(-4, -3)	(-3, 0)	(0, 2)	(2, ∞)
sign of x	—	—	—	+	+
sign of x - 2	—	—	—	—	+
sign of (x + 3)(x+3)	+	+	+	+	+
sign of x + 4	—	+	+	+	+
sign of x(x-2)(x+3)(x+3)(x+4)	—	+	+	—	+

Fig. 7

The final row of this table shows where the polynomial is positive and where it is negative. We therefore conclude that:
The solution set for x(x – 2)(x + 3)(x + 3)(x + 4) = 0 is $\{-4,-3,0,2\}$.
The solution set for x(x – 2)(x + 3)(x + 3)(x + 4) > 0 is $(-4,-3)\cup(-3,0)\cup(2,\infty)$.
The solution set for x(x – 2)(x + 3)(x + 3)(x + 4) < 0 is $(-\infty,-4)\cup(0,2)$.

Important Notes About Polynomial Inequalities: As a first step when working with polynomial inequalities one should write the inequality in the form

$$a_n x^n + a_{n-1}x^{n-1} + \bullet\bullet\bullet + a_1 x + a_0 < 0 \text{ or}$$
$$a_n x^n + a_{n-1}x^{n-1} + \bullet\bullet\bullet + a_1 x + a_0 > 0$$

Writing a polynomial inequality in the desired form may be done using the three familiar, fundamental, elementary properties of inequalities.

(1): If any expression is added to both sides of an inequality the resulting inequality is equivalent to the original inequality.

(2): If both sides of an inequality are multiplied by the same positive real number, the resulting inequality is equivalent to the original inequality.

(3): If both sides of an inequality are multiplied by the same negative real number and the inequality symbol is reversed, the resulting inequality is equivalent to the original inequality.

As a first step when working with polynomial inequalities one should find the real solutions of the corresponding boundary equation and factor the polynomial into a product of linear and quadratic factors. Although this is theoretically always possible, it is generally very difficult and frequently impossible to do. When the real solutions of the boundary equation cannot be found, other methods, beyond elementary algebra, must be employed.

A table such as shown in Fig. 7 is especially useful and almost essential when the expression involved is complicated.

Law of Trichotomy Applied to Absolute Value Inequalities in One Variable

For the sake of simplicity, the following discussion of absolute value equations and inequalities will be restricted to absolute values of linear expressions. Much of this is true more generally, but this discussion is explicitly for Intermediate and College Algebra classes. Extensions can easily be made in other courses when necessary.

Recall that earlier in this essay it was pointed out that the graph of the equation is a boundary between the graphs of the corresponding inequalities. For that reason, the equation is sometimes called the boundary equation for the inequalities. We will use that terminology.

Recall the definition of absolute value.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Any discussion of equations and inequalities involving absolute values must begin by noting that we are discussing equations and inequalities of the form

$|M| < k$ $\quad\quad$ $|M| = k$ $\quad\quad$ $|M| > k$

where M is some linear expression and k is a real number.

Observe the Law of Trichotomy dictates that $k < 0$, $k = 0$, or $k > 0$. The three possibilities for k combined with the boundary equation and the two inequalities forces us to consider nine cases. Fortunately most of them are trivial and can be dealt with quite easily. We will separately consider the three possibilities ($k = 0$, $k < 0$, and $k > 0$) for the number k.

Possibility 1: k = 0.

Case 1: (The boundary equation) If M is an algebraic expression, then $|M| = k$ has the form $|M| = 0$ which is equivalent to $M = 0$.

To understand this look carefully at the definition of absolute value and you will observe that the absolute value of a quantity is 0 if and only if the quantity is 0. Absolute value inequalities of the form $|M| = 0$ are therefore equivalent to the equality obtained by removing the absolute value symbol. That equality is easily solved using the normal method for solving linear equations in one variable. For anyone who has forgotten this normal method here it is in a nutshell.

The process to solve a linear equation in one variable is to use the following two properties of equations to generate a sequence of equations each equivalent to the previous equation until a simplest equation is obtained.

Properties of Equations:

(1): If any expression is added to both sides of an equation the resulting equation is equivalent to the original equation.

(2): If both sides of an equation are multiplied by the same non-zero real number, the resulting equation is equivalent to the original equation.

The following four examples illustrate the solving process when the equality is of the form |M| = 0.

Example: |x| = 0 is equivalent to the simplest equation x = 0 whose solution set is {0}.

Example: |3x – 8| = 0 is equivalent to 3x – 8 = 0 which in turn is equivalent to the simplest equation $x = \frac{8}{3}$ whose solution set is $\left\{\frac{8}{3}\right\}$.

Example: |–5x + 7| = 0 is equivalent to –5x + 7 = 0 which in turn is equivalent to the simplest equation $x = \frac{7}{5}$ whose solution set is $\left\{\frac{7}{5}\right\}$.

Example: |–2x – 6| = 0 is equivalent to –2x – 6 = 0 which in turn is equivalent to the simplest equation x = –3 whose solution set is {–3}.

Case 2: (The "less than" inequality) If M is an algebraic expression, then |M| < k has the form |M| < 0 whose solution set is the null set. The absolute value cannot be negative!

Example: |3x – 7| < 0 has no solution. The absolute value of 3x – 7 cannot be negative.

Case 3: (The "greater than" inequality) If M is an algebraic expression, then the solution set for |M| > k has the form |M| > 0 whose solution set is the set of all real numbers except for the solutions of M = 0.

Example: The solution set for |3x – 9| > 0 is all real numbers except 3. The absolute value of 3x – 9 is positive except when x = 3, in which case the absolute value is 0. Therefore the solution set for |3x – 9| > 0 is the set of all real numbers except 3.

Possibility 2: k < 0.

Case 1: (The boundary equation) If M is an algebraic expression and k is negative, then the solution set for |M| = k is the null set. The absolute value cannot be negative!

Example: The solution |–2x – 6| = –3 is the null set because the absolute value is never negative.

Case 2: (The "less than" inequality) If M is an algebraic expression and k is negative, then the solution set for |M| < k is the null set. The absolute value cannot be negative!

Example: The solution |–2x + 1| < –5 is the null set because the absolute value is never negative.

Case 3: (The "greater than" inequality) If M is an algebraic expression, then the solution set for |M| > k is the set of all real numbers.

Example: The solution $|5x - 20| > -3$ is the all real numbers because the absolute value of $5x - 20$ is non-negative for all values of x.

Possibility 3: k > 0.

This is the only nontrivial situation about equations and inequalities involving absolute value. The classic approach is to teach three similar and closely related but distinct methods for solving the boundary equation and the two inequalities (=, <, >). Those methods are each based on the two cases implied by the definition of absolute value. The classic method usually involves a somewhat confusing (for beginning students) use of set intersection and union.

The Law of Trichotomy, and one other detail, provides an alternate and much easier single method for simultaneously solving the boundary equation and both inequalities.

To better understand the role of the Law of Trichotomy, consider the following discussion of a numerical example.
Consider the inequalities $|2x + 5| < 1$, $|2x + 5| > 1$, and the boundary equation $|2x + 5| = 1$. If we let $x = 4$, then the three statements become $|8 + 5| < 1$, $|8 + 5| > 1$, and $|8 + 5| = 1$. Even without any computations we know from the Law of Trichotomy that exactly one of these statements is true. Or stated slightly differently, the Law of Trichotomy tells us that 4 is a solution of exactly one of the two inequalities $|2x + 5| < 1$, $|2x + 5| > 1$, or the boundary equation $|2x + 5| = 1$. This same argument can be repeated for any real number choice for x, so the inevitable conclusion is that each real number is a solution of exactly one of the inequalities $|2x + 5| < 1$, $|2x + 5| > 1$, or the boundary equation $|2x + 5| = 1$. There is nothing special about this example, so we conclude the following general principle.

Each real number is a solution of exactly one of $|M| < k$, $|M| = k$, or $|M| > k$ where M is some linear expression and k is a positive real number.

Another very useful way of stating this is:

Fact A: The union of the solution sets of $|M| < k$, $|M| = k$, and $|M| > k$ is the real numbers and the intersection of any two of these solution sets is the empty set.

It is a fact that $|M| < k$ is easy to solve and the solution set for the other two $|M| = k$, and $|M| > k$ can be deduced from the solution set of $|M| < k$. The following example should help you to understand that statement.

Consider $|2x + 5| < 7$. From a very early understanding of absolute value $|2x + 5|$ means the distance $2x + 5$ is from 0. Therefore the statement $|2x + 5| < 7$ means the distance between 0 and $2x + 5$ is less than 7. It is clear that $2x + 5$ must be between -7 and $+7$. It is more useful to think $2x + 5$ must be between 7 and its opposite. That is $-7 < 2x + 5$ AND $2x + 5 < 7$.
This compound inequality is usually written using the compact form $-7 < 2x + 5 < 7$.
The solution set for $|2x + 5| < 7$ is found by solving $-7 < 2x + 5 < 7$ to obtain the interval $(-6, 1)$.
There is nothing special about this example so we can jump to the following general statements.

Fact B: If k is a positive real number and M is a linear expression, then |M| < k is equivalent to the compound inequality – k < M < k.

Example: The inequality |2x – 3| < 7 is equivalent to –7 < 2x – 3 < 7.

Example: The inequality |5x – 8| < 3 is equivalent to -3 < 5x – 8 < 3.

Example: The inequality $\left|\frac{2}{3}x-\frac{7}{5}\right|<\frac{\sqrt{2}}{\pi}$ is equivalent to $-\frac{\sqrt{2}}{\pi}<\frac{2}{3}x-\frac{7}{5}<\frac{\sqrt{2}}{\pi}$

Observe that in the general statement and each of the examples the expression inside the absolute value symbol is wedged between the number k and its opposite.

Fact C: If k is a positive real number and M is a linear expression, then the solution set for |M| < k is an interval on the real number line.

Example: The inequality |2x – 3| < 7 is equivalent to –7 < 2x – 3 < 7 which is equivalent to –4 < 2x < 10 which is equivalent to –2 < x < 5 whose solution is the interval (–2, 5).

It follows that:

Fact D: The endpoints of that interval are the solutions of the boundary equation |M| = k.

Example: (–2, 5) is the solution set for |2x – 3| < 7. Its endpoints are the solutions of the corresponding boundary equation |2x – 3| = 7. The solution set for the boundary equation |2x – 3| = 7 is {–2, 5}.

And finally it follows that:

Fact E: Every real number outside the interval and not an endpoint of the interval is a solution to |M| > k.

Example: (–2, 5) is the solution set for |2x – 3| < 7. Its endpoints are the solutions of the corresponding boundary equation |2x – 3| = 7. The solution set for the boundary equation |2x – 3| = 7 is {–2, 5}. The remainder of the real number line must be the solution set for the "greater than" inequality. The solution set for |2x – 3| > 7 is $(-\infty,-2)\cup(5,\infty)$.

Compact compound inequalities of the form – k < M < k as mentioned in Fact B are easy to solve. Such an inequality may always be solved using only the three familiar, fundamental, elementary properties of inequalities. That process is summarized here.

The process to solve a linear inequality in one variable is to use the following three properties of inequalities to generate a sequence of inequalities each equivalent to the previous inequality until a simplest inequality is obtained.

Properties of Inequalities:

(1): If any expression is added to both sides of an inequality the resulting inequality is equivalent to the original inequality.

(2): If both sides of an inequality are multiplied by the same positive real number, the resulting inequality is equivalent to the original inequality.

(3): If both sides of an inequality are multiplied by the same negative real number and the inequality symbol is reversed, the resulting inequality is equivalent to the original inequality.

If all of the above observations (Facts A, B, C, D, and E) are put together we arrive at the following single easy process for solving any one (and in fact all) of $|M| < k$, $|M| = k$, or $|M| > k$ where M is some linear expression and k is a positive real number

The Process:

Step 1: Regardless of the question being asked, focus on the "less–than" inequality.

Step 2: Convert that inequality to the equivalent compound inequality.

Step 3: Solve the compound inequality.

Step 4: Write the solution set and graph it (at least visualize it) on the Real Number Line.

Step 5: Identify the endpoints of this solution set as the elements of the solution set for the corresponding boundary equation.

Step 6: Identify the remainder of the Real Number line as the solution set for the "greater–than" inequality.

It is worth observing again that this approach always explicitly produces the solutions set to all three possibilities (<, =, >).

The discussion will be finalized with some examples.
Example: Solve the inequality $|3x + 5| < 2$.
Discussion and Solution:
We begin by converting $|3x + 5| < 2$ to the equivalent compound inequality by wedging $3x + 5$ between 2 and its opposite.

$-2 < 3x + 5 < 2$ add –5 to all sides

$-7 < 3x < -3$ multiply both sides by $\frac{1}{3}$

$-\frac{7}{3} < x < -1$

The solution set for $|3x + 5| < 2$ is the interval $\left(-\frac{7}{3}, -1\right)$.

If set builder notation is used to write this solution set we have $\left\{x \middle| -\frac{7}{3} < x < -1\right\}$.

The graph of this solution set is shown in Fig. 8.

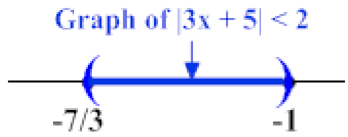

Fig. 8

As an easy bonus we observe that no further computation is required to deduce the solution sets for |3x + 5| = 2 and |3x + 5| > 2

The solution of the boundary equation |3x + 5| = 2 consists of the endpoints of the interval $\left(-\frac{7}{3}, -1\right)$. Therefore the solution set for |3x + 5| = 2 is the $\left\{-\frac{7}{3}, -1\right\}$.

The graph of this solution set is Fig. 9.

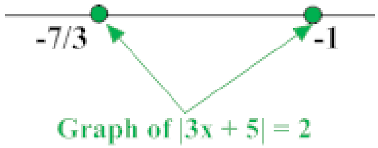

Fig. 9

Furthermore we can conclude that the solution set for the “greater than” inequality |3x + 5| > 2 is everything else. The solution set for |3x + 5| > 2 is $\left(-\infty, -\frac{7}{3}\right) \cup (-1, \infty)$.

The graph of this solution set is in Fig. 10.

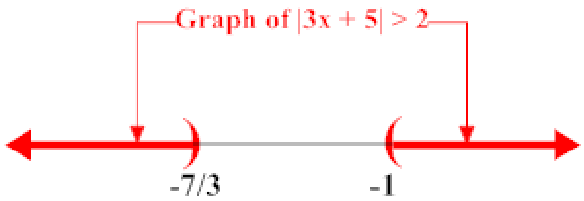

Fig. 10

It is instructive to graph all three solution sets on the same number line to obtain

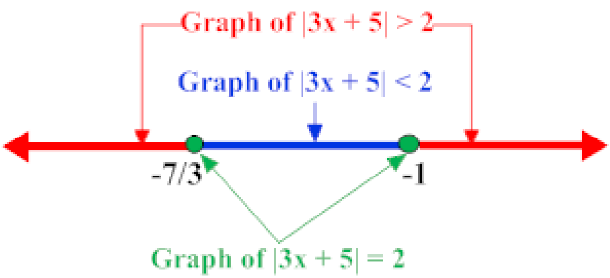

Fig. 11

and observe that the solution sets do not overlap and their union is the entire real number line.

Example: Solve the equality $\left|3x - \frac{5}{8}\right| = 1$.

Discussion and Solution:
We begin by focusing not on the given equality, but on the "less than" inequality $\left|3x - \frac{5}{8}\right| < 1$ which we immediately convert to the equivalent compound inequality and solve with normal techniques.

$-1 < 3x - \frac{5}{8} < 1$ add $\frac{5}{8}$ to all sides

$-\frac{3}{8} < 3x < \frac{13}{8}$ multiply all sides by $\frac{1}{3}$

$\left(\frac{1}{3}\right)\left(-\frac{3}{8}\right) < x < \left(\frac{1}{3}\right)\left(\frac{13}{8}\right)$ simplify

$-\frac{1}{8} < x < \frac{13}{24}$

The solution set for the inequality $\left|3x - \frac{5}{8}\right| < 1$ is the interval $\left(-\frac{1}{8}, \frac{13}{24}\right)$.
The graph is show in Fig. 12.

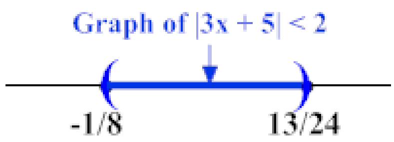

Fig. 12

With no further computations we can deduce the solutions for the desired original equality $\left|3x - \frac{5}{8}\right| = 1$ are the endpoints $-\frac{1}{8}$ and $\frac{13}{24}$.
The solution set for $\left|3x - \frac{5}{8}\right| = 1$ is $\left\{-\frac{1}{8}, \frac{13}{24}\right\}$.
The graph is shown in Fig. 13.

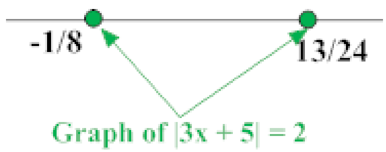

Fig. 13

As a bonus we also know the solution set for the "greater than" inequality is everything else and is therefore $\left(-\infty, -\frac{1}{8}\right) \cup \left(\frac{13}{24}, \infty\right)$.
The graph is shown in Fig. 14.

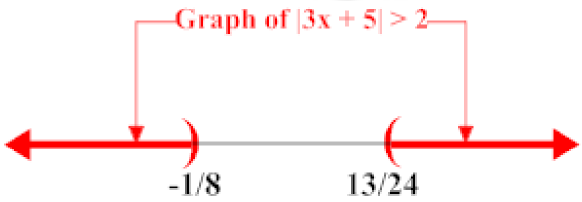

Fig. 14

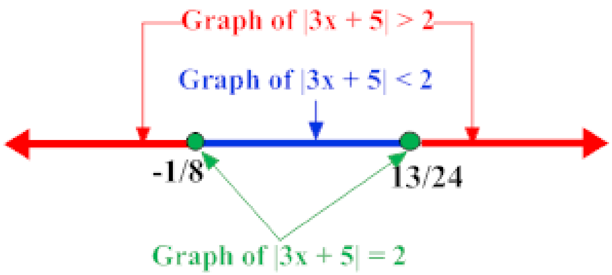

Fig. 15

Again it is instructive to graph all three solution sets on the same number line to obtain Fig. 15 and observe that the solution sets do not overlap and their union is the entire real number line.

Example: Solve the inequality |3x – 5| > 6.
Discussion and Solution:

We begin by focusing not on the given inequality, but on its counterpart $|3x - 5| < 6$ which we immediately convert to the equivalent compact compound inequality.

$-6 < 3x - 5 < 6$ add 5 to both sides

$-1 < 3x < 11$ multiply both sides by $\frac{1}{3}$

$-\frac{1}{3} < x < \frac{11}{3}$

The solution set for $|3x - 5| < 6$ is the interval $\left(-\frac{1}{3}, \frac{11}{3}\right)$.

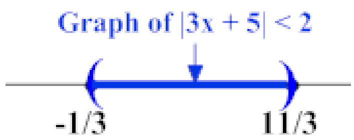

Fig. 16

The graph is shown in Fig. 16.
No further computation is required to deduce the solution sets for $|3x - 5| = 6$ and $|3x - 5| > 6$

The solution set for $|3x - 5| = 6$ is $\left\{-\frac{1}{3}, \frac{11}{3}\right\}$ whose graph is Fig. 17.

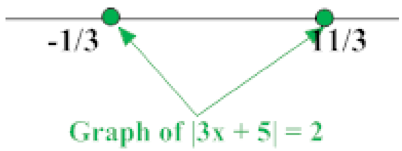

Fig. 17

The solution set for the <u>desired</u> inequality $|3x - 5| > 6$ is

$\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{11}{3}, \infty\right)$.

The graph is shown in Fig. 18.

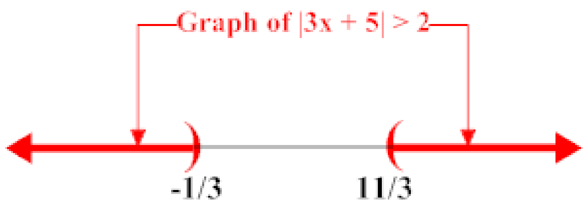

Fig.18

As before we can graph all three solution sets on the same number line to obtain Fig. 19

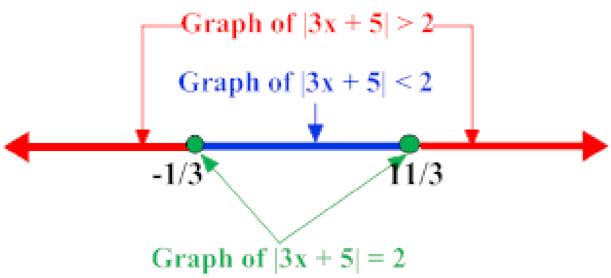

Fig. 19

Where it is easy to observe that the solution sets do not overlap and their union is the entire real number line.

Law of Trichotomy Applied to Equations and Inequalities in Two Variables

Linear Equations and Inequalities in Two Variables

A **linear equation in two variables** is an equation which may be written in the form Ax + By = C where A, B, and C are real numbers and B is not zero.

A **linear inequality in two variables** x and y is an inequality which can be written as Ax + By < C, Ax + By > C.

Law of Trichotomy: If a and b are real numbers, then one and only one of the following is true:

1. a < b
2. a = b
3. a > b

The Law of Trichotomy affects how we consider equations and inequalitites in two variables. When considering any one of the three possibilities Ax + By = C, Ax + By < C, or Ax + By > C, we consider all three.

The graph of a linear equation in two variables is a line in the coordinate plane which divides the coordinate plane into two half. One of those half-planes is the graph of Ax + By < C, and the other half-plane is the graph of Ax + By > C.

When asked to analyze either of Ax + By = C, Ax + By < C, or Ax + By > C, we begin by graphing the equation by plotting two points (usually the x and y intercepts) and then testing one point (not on the boundary line) in one of the inequalities. Those few simple steps provide us with a complete analysis of the equation and both inequalities. The equation is called the **boundary equation** because its graph forms a boundary between the graphs of the two inequalities.

The following will illustrate the process and the results.
EXAMPLE: Discuss the inequality 3x – 7y < 21.
Begin by analyzing the equation of the boundary line is 3x – 7y = 21.
This is a linear equation in two variables so its graph is a line in the rectangular coordinate system.
If x = 0, then y = –3, so the point (0, –3) is the y-intercept of the boundary line.
If y = 0, then x = 7, so the point (7, 0) is the x-intercept of the boundary line.
Plot those two points and draw the line through them to obtain the graph of the boundary line as shown in green in Fig 20.

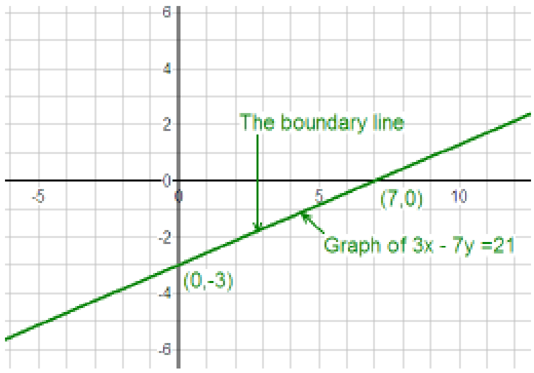

Fig 20

The origin is not on the graph of the boundary line and is easy to test in either of the inequalities.
Use (0, 0) as a test point in the inequality 3x – 7y < 21 to obtain
3(0) – 7(0) < 21 which is TRUE.
Therefore (0, 0) is a solution of 3x – 7y < 21 and every other point in that half-plane is a solution of the inequality 3x – 7y < 21.
Conclusion: The solution set for 3x – 7y < 21 is the half-plane containing the origin and bounded by the graph of 3x – 7y = 21.
We graph the inequality by shading the half-plane which is its solution set as shown in blue in Fig. 21.

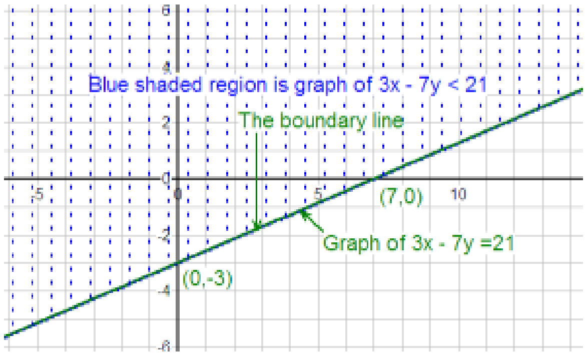

Fig. 21

Because of the Law of Trichotomy we know that the other half-plane (the un-shaded part) is the graph of 3x – 7y > 21.
Conclusion: The solution set for 3x – 7y > 21 is the half-plane NOT containing the origin and bounded by the graph of 3x – 7y = 21.
For consistency we show this graph in red in Fig. 22.

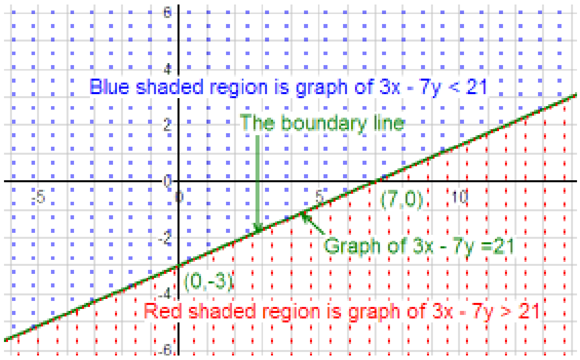

Fig. 22

EXAMPLE: Discuss the system of inequalities $\begin{cases} 2x + 5y \geq 10 \\ -3x + 2y \geq 6 \\ 2x - 7y > -38 \end{cases}$

Begin by observing that this is a system of three linear inequalities in two variables. The solution set for each individual inequality is a half-plane. (The Law of Trichotomy).
The solution set for the system is the set of points which are solutions of 2x + 5y ≥ 10 AND –3x + 2y ≥ 6 AND 2x – 7y > – 38. Therefore the solution set of the system is the intersection of the three individual solution sets.
A good strategy is to graph each of the individual inequalities in different colors on the same coordinate system so the intersections is easily observed to be the region shaded in all three colors. For instructional purposes, each inequality will initially be graphed on its own coordinate system and then they will be combined.

The boundary equation for 2x + 5y ≥ 10 is 2x + 5y = 10.
If x = 0, then y = 2. Therefore (0, 2) is the y-intercept of the boundary equation.
If y = 0, then x = 5. Therefore (5, 0) is the x-intercept of the boundary equation.

The origin is not on the boundary line so we test it in 2x + 5y ≥ 10 to obtain 0 ≥ 10 which is FALSE.
Therefore the graph of 2x + 5y ≥ 10 is the half-plane which does not contain the origin and is bounded by the boundary line 2x + 5y = 10. Because the symbol ≥ permits equality, the boundary line is part of the solution set and is therefore drawn as a solid line to indicate that it is part of the graph.
The graph of 2x + 5y ≥ 10 appears in Fig. 23.

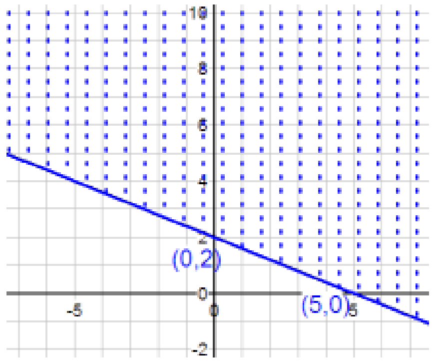

Fig. 23

The boundary equation for –3x + 2y ≥ 6 is –3x + 2y = 6.
If x = 0, then y = 3. Therefore (0, 3) is the y-intercept of the boundary equation.
If y = 0, then x = –2. Therefore (–2, 0) is the x-intercept of the boundary equation.
The origin is not on the boundary line so we test it in –3x + 2y ≥ 6 to obtain 0 ≥ 6 which is FALSE.
Therefore the graph of –3x + 2y ≥ 6 is the half-plane which does not contain the origin and is bounded by the boundary line –3x + 2y = 6. Because the symbol ≥ permits equality, the boundary line is part of the solution set and is therefore drawn as a solid line to indicate that it is part of the graph.
The graph of –3x + 2y ≥ 6 appears in Fig. 24.

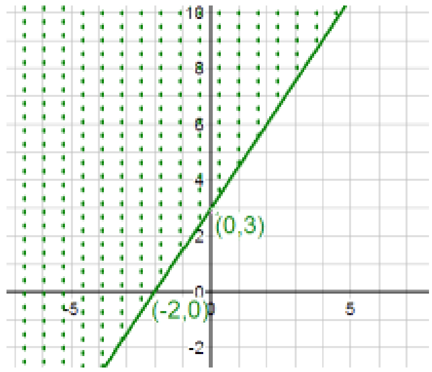

Fig. 24

The boundary equation for 2x – 7y > – 38 is 2x – 7y = – 38.
If x = 0, then $y = \frac{38}{7}$. Therefore $\left(0, \frac{38}{7}\right)$ is the y-intercept of the boundary equation.
If y = 0, then x = –19. Therefore (–19, 0) is the x-intercept of the boundary equation.
The origin is not on the boundary line so we test it in 2x – 7y > – 38 to obtain 0 ≥ –38 which is TRUE.
Therefore the graph of 2x – 7y > – 38 is the half-plane which contains the origin and is bounded by the boundary line 2x – 7y = – 38. Because the symbol > does not permit equality, the boundary line is not part of the solution set and is therefore drawn as a dashed line to indicate that it is not part of the graph.
The graph of 2x – 7y > – 38 appears in Fig. 25.

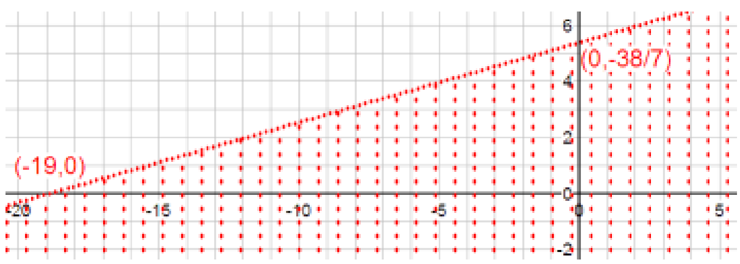

Fig. 25

Recall that the graph of the system is the intersection of the graphs of the individual inequalities in the system. In Fig. 26 the three graphs are superimposed on the same coordinate system and their intersection becomes clear.
The graph of the system of inequalities is shown in Fig. 26.
The solution of the system

$$\begin{cases} 2x + 5y \geq 10 \\ -3x + 2y \geq 6 \\ 2x - 7y > -38 \end{cases}$$

is the triangular region containing red, blue, and green shading and bounded by the three boundary lines. Note the blue and green boundaries of the triangle are part of the graph and the red boundary is not part of the graph.

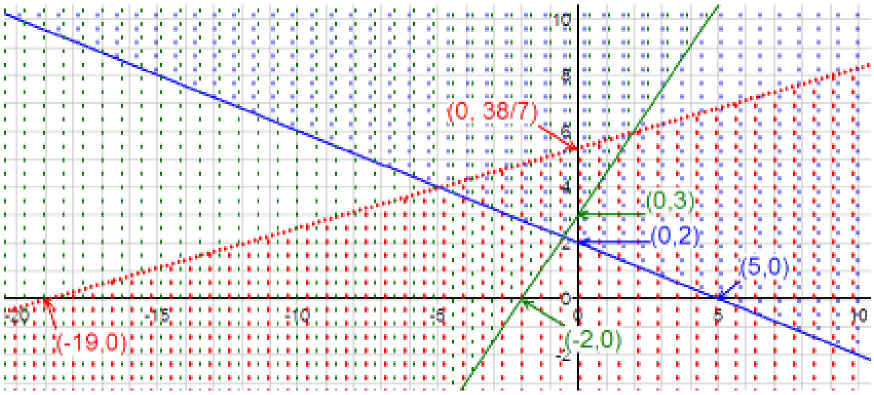

Fig. 26

This graph tells us that every point in the triangle or on the red or on the blue edge of the triangle is a solution of the system of inequalities.
In terms of sets and unions we see that the solution set of the system is the set of points in the interior of the triangle UNION the set of points on the blue boundary of the triangle UUNION the set of points on the green boundary of the triangle.