Analysis of Rational Functions

Question:

Analyze and graph the rational function whose rule is $f(x) = \frac{2x-3}{x-4}$.

Analysis:

a) Determine the domain of f

When the domain of a function is not explicitly stated, convention dictates that the domain is the largest subset of the real numbers for which the rule make sense (is defined).

In the case of a rational function the rule for f is defined for all real numbers except for zeros of the denominator.

In this case the rule for f is defined for all real numbers except when x – 4 = 0.

The domain of f is all real numbers except 4.

The domain of f is described with set-builder notation as $D_f = \{x \mid x \in \mathbf{R}, x \neq 4\}$.

The domain of f is described with interval notation as $D_f = (-\infty, 4) \cup (4, +\infty)$

Figure 1 summarizes this work.

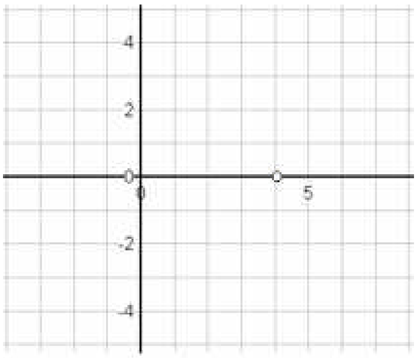

Figure 1

b) Determine the zeros of f

The zeros of any function f are found by solving the equation resulting from f(x) =0.

In this case we must solve the equation

Equation 1: $\frac{2x-3}{x-4} = 0$.

Multiply both sides of the equation by the denominator x - 4 to obtain

Equation 2: 2x – 3 = 0.

Observe that Equation 2 need not be equivalent to Equation1. However, the solution set for Equation 2 contains the solution set for Equation 1. Therefore all solutions of Equation 2 are potential solutions of Equation 1, but it is necessary to test solutions of Equation 2 to determine if they are solutions of Equation 1. In the case of a rational equation, it is only necessary to insure that a potential solution not cause a 0 in a denominator.

The solution set for Equation 2 is $\left\{\frac{3}{2}\right\}$ and clearly $\frac{3}{2}$ does not cause a 0 in the denominator of f.

The zero of f is $\frac{3}{2}$.

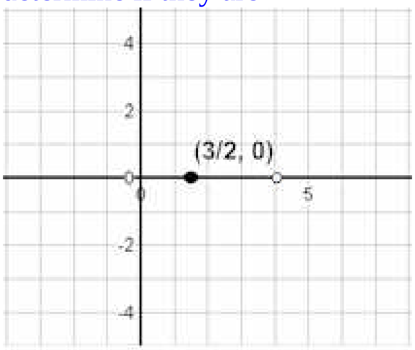

Figure 2

c) Determine the vertical asymptotes of f

Vertical asymptotes occur at zeros of the denominator which are not zeros of the numerator. The zero of the denominator is 4, which is not equal to $\frac{3}{2}$, the zero of the numerator. Therefore the vertical line x = 4 is a vertical asymptote of the graph of the function f.

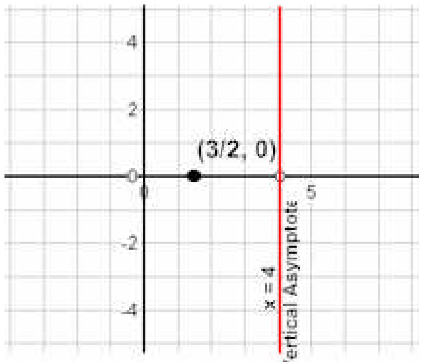

Figure 3

d) Determine and shade excluded regions

Begin by sketching the vertical asymptotes on a rectangular coordinate system. Then sketch a vertical line through each zero of the function. The lines through the zeros of f are construction lines and are not part of the graph of the function f. These vertical lines should be removed from the final graph of the function f. However, the vertical asymptotes are frequently included with the final graph and are frequently considered to be part of the graph.

The vertical lines through the zeros of f combined with the vertical asymptotes of f divide the coordinate system into a number of strips. In each individual strip the graph of f will either be entirely above the x-axis or entirely below the x-axis. Test a single convenient domain element k in each strip to determine whether f(k) > 0 or f(k) < 0.
If f(k) > 0, the graph of f will be above the x-axis in that strip and the portion of that strip which is below the x-axis is called an "excluded region" because the graph is excluded form that region. Shade all excluded regions.
If f(k) < 0, the graph of f will be below the x-axis in that strip and the portion of that strip which is above the x-axis is called an "excluded region" because the graph is excluded form that region. Shade all excluded regions.

In this example:

Test 0: $f(0) = \frac{2x-3}{x-4} = \frac{2(0)-3}{0-4} = \frac{-3}{-4} > 0$.

f(0) > 0 is the important observation.

Therefore the half-strip to the left of $\frac{3}{2}$ and below the x-axis is excluded.

Test 3: $f(3) = \frac{2(3)-3}{3-4} = \frac{3}{-1} = 3 > 0$

f(3) < 0 is the important observation.

Therefore the half-strip between $\frac{3}{2}$ and 4 and above the x-axis is excluded.

Test 5: $f(5) = \frac{2(5)-3}{5-4} = \frac{7}{1} = 7 > 0$

f(5) > 0 is the important observation.

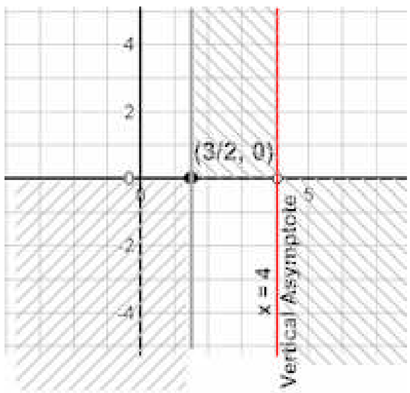

Figure 4

e) Determine the horizontal asymptotes of f

The ratio of the degrees of the numerator and denominator of a rational function determine whether the function does or does not have a horizontal asymptote.

1. If the degree of the numerator is greater than the degree of the denominator, the function does not have a horizontal asymptote.
2. If the degree of the numerator is equal to the degree of the denominator, the function has a horizontal asymptote $y = \frac{a_m}{b_n}$ where a_m is the leading coefficient of the numerator and b_n is the leading coefficient of the denominator.
3. If the degree of the numerator is less than the degree of the denominator, the function has the x-axis as its horizontal asymptote.

The numerator and denominator of the function f in this example have the same degree, so the horizontal asymptote is the horizontal line $y = \frac{2}{1} = 2$.

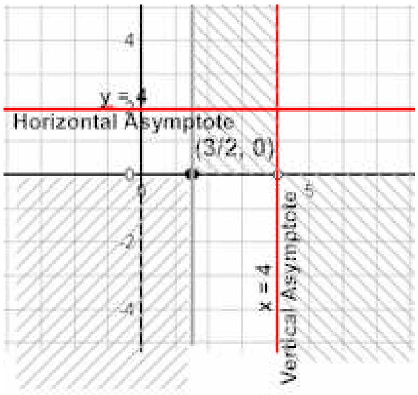

Figure 5

The horizontal asymptote is y = 2.

f) Determine where (if at all) the graph of f intersects its horizontal asymptote

The graph of f intersects its horizontal asymptote if and only if f(x) = 2 for some x. Thus we find the solutions of the equation $\frac{2x-3}{x-4} = 2$. Multiply both sides of the equation by x - 4 to obtain 2x – 3 = 2x – 8. Add -2x to both sides of the equation to obtain – 3 = – 8; a contradiction.

Therefore the equation 2x – 3 = 2x – 8 has no solution and since the solution set for the equation $\frac{2x-3}{x-4} = 2$ is contained in the solution set for 2x – 3 = 2x – 8 the original equation $\frac{2x-3}{x-4} = 2$ has no solution and the graph of f does not intersect its horizontal asymptote.

The graph of f does not intersect its horizontal asymptote.

g) Sketch the graph of f

The graph of f as generated by a computer graphing utility is shown in Figure 6. The reader should verify that this graph matches the preceding analysis.

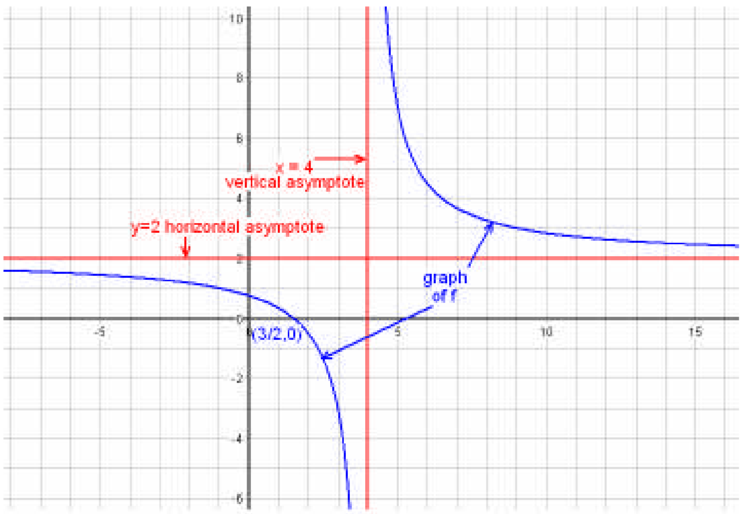

Figure 6

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h) Determine where the graph of f has horizontal tangents.

i) Determine where the graph of f is increasing, decreasing.

j) Determine where the graph of f has relative maxima and/or relative minima.

k) Determine where the graph of f is concave down, up.

l) Modify the graph of f as indicated by the calculus investigations .